Commun. Math. Phys. 325, 259–289 (2014)
Digital Object Identifier (DOI) 10.1007/s00220-013-1859-y

Communications in **Mathematical Physics**

Trees of Nuclei and Bounds on the Number of Triangulations of the 3-Ball

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Received: 30 April 2012 / Accepted: 12 June 2013
Published online: 24 December 2013 –

Abstract: Based on the work of Durhuus–Jónsson and Benedetti–Ziegler, we revisit the question of the number of triangulations of the 3-ball. We introduce a notion of nucleus (a triangulation of the 3-ball without internal nodes, and with each internal face having at most 1 external edge). We show that every triangulation can be built from trees of nuclei. This leads to a new reformulation of this question: We show that if the number of rooted nuclei with t tetrahedra has a bound of the form C^t, then the number of rooted triangulations with t tetrahedra is bounded by C_*^t.

Contents

1. Introduction

In this paper, we study the question of the number of triangulations of the 3-ball by tetrahedra. The case of the 2-ball was exactly solved by Tutte in [17]. He showed in particular that the number of rooted triangulations of the 2-sphere with N vertices is $\mathcal{O}(1)\, N^{-5/2}(256/27)^N$. It is natural to ask if analogous bounds are true in higher dimension. Such results could have applications in models of Statistical Mechanics (foams [15], quantum gravity [3], or glassy dynamics [1,4,9,10]) where the exponential rate of growth can be interpreted as an entropy. The problem of the existence of an exponential bound in 3-dimensions was formulated by Ambjørn, Durhuus and Jónsson in [2] and emphasized by Gromov in [13]: they asked whether the number of triangulations of the 3-sphere is bounded by C^N for some constant C when there are N tetrahedra (facets) in the triangulation. To date, this question remains open. However Pfeifle and Ziegler proved in [16] a super exponential lower bound for the number of triangulations of the 3-ball as a function of the number of vertices. This does not answer negatively Gromov's question (which is in terms of the number of tetrahedra) but makes the problem of proving an exponential bound in terms of the number of tetrahedra even more challenging.

There are several studies in the direction of answering the question, which we summarize now. In [8], Durhuus and Jónsson gave the construction of a class of triangulations for which they could show a bound of the form C^N. These triangulations are obtained by building a tree of tetrahedra, which is obtained by starting from a root tetrahedron and attaching tetrahedra to its faces, and then attaching further tetrahedra to the new open faces. Each tetrahedron is attached to the tree with just one face. It is a common feature of tree-like constructions that they lead to bounds of the form C^N: The prime example in our context is of course the celebrated work of Tutte [17] mentioned above. Coming back to Durhuus and Jónsson, once the tree is constructed, they now collapse adjacent faces of the tree in such a way that at the end of the procedure a triangulation of the 3-sphere is obtained. Their main result says that the number of ways in which to do this is again exponentially bounded. In this way, they construct a set of triangulations of the 3-sphere with tetrahedra which is exponentially bounded. They ask whether these are all possible triangulations.

In a later development, Benedetti and Ziegler [5], show that the Durhuus and Jónsson construction, which they call "locally constructible" (LC), does *not* capture all triangulations of the 3-sphere. Namely, they show that a 3-sphere with a 3-complicated knotted edge (made by tetrahedra) is not LC. They also carefully discuss relations between LC and other classes of constructibility.

In the present paper, we define a larger class of triangulations, with a construction similar to that of Durhuus and Jónsson, but which uses more general basic elements than the simple tetrahedron, which we call *nuclei*. We prefer to work with 3-balls, and bounds on 3-spheres can be obtained from a bound on triangulations of a tetrahedron. This is usually done by removing a tetrahedron from the 3-sphere (see for example [5, Sect. 3]).

Nuclei are defined as triangulations of the 3-ball with the following special properties:

1. They have no internal nodes.
2. Internal faces have at most *one* external edge.

Obviously, the tetrahedron is a nucleus. The Furch–Bing ball [6,11 and 14] and the Bing 2-room house [6] and [14], which are not nuclei, can be reduced by our procedure to one non-trivial nucleus, each. The smallest non-trivial nucleus we know of, given in Table 1, has 12 nodes, and 37 tetrahedra, of which 17 have no external face. Nodes are numbered from 1 to 12, and Table 1 gives a list of the 37 tetrahedra.

Table 1. A nucleus with 12 nodes, and 37 tetrahedra, of which 17 have no external face

1	**3**	**4**	10	1	3	5	10	**1**	**3**	5	**11**	**1**	**4**	**6**	10	1	5	7	8
1	**5**	**7**	**10**	**1**	5	**8**	**11**	**1**	**6**	7	**8**	1	6	7	10	**2**	3	**5**	**9**
2	3	5	11	2	3	8	9	2	**3**	**8**	**11**	**2**	**5**	**6**	11	**2**	**6**	11	**12**
2	**7**	**10**	11	**2**	**7**	11	**12**	**2**	8	**9**	**10**	2	8	10	11	3	4	9	10
3	**4**	9	**12**	3	**5**	**9**	**10**	**3**	**8**	9	**12**	**4**	**5**	**6**	11	**4**	**5**	**7**	8
4	5	8	11	4	6	10	11	4	7	8	9	**4**	**7**	9	**12**	4	8	9	10
4	8	10	11	6	7	8	9	6	7	9	11	6	7	10	11	**6**	**8**	9	**12**
6	9	11	12	7	9	11	12												

If a tetrahedron has an external face, its 3 nodes are shown in boldface

Table 2. Experimental upper bound on the number of cones needed to decompose a triangulation into tetrahedra (for the definition of m-complicated, see [5])

Example	Knot complication	# of cones added	Ref.
Bing 2 room	No knot	1 cone	[6]
1 trefoil	1-complicated	1 cone	[11]
2 trefoils	2-complicated	1 cone	
3 trefoils	3-complicated	2 cones	[5, Figure 3]
4 trefoils	4-complicated	3 cones	
5 trefoils	5-complicated	3 cones	
Figure eight	1-complicated	1 cone	
Cinquefoil knot	1-complicated	1 cone	

Our approach is twofold: Top-down, and bottom-up. In the top-down approach, we define a set of elementary moves which reduce an arbitrary triangulation of the 3-ball into a tree of nuclei, which are glued together by pairs of faces, each such face with 3 external edges. The tree can then be cut into a disjoint union of nuclei by cutting along these faces. The construction always transforms 3-balls to unions of 3-balls, and is thus implementable on a computer.

In the bottom-up approach, we start with any tree whose nodes are arbitrary nuclei, and we construct 3-balls from it by gluing adequate faces together. Not all possible gluings lead to 3-balls, but including also some inadequate gluings still leads to good bounds. Again, the procedure can be programmed on a computer.

Our main result is Theorem 5.17. It says that *if the number* $\varrho(t, f_{\mathrm{s}})$ *of face-rooted nuclei with* t *tetrahedra and* f_{s} *external faces has a bound of the form* $\varrho(t, f_{\mathrm{s}}) \le C^t$ *then the number of rooted triangulations of the 3-ball with* t *tetrahedra,* f *external faces and* n *internal nodes is bounded by* C_*^{t+f+n}.

In particular, since obviously, $f \le 4t$ and $n \le 4t$, we would get a bound C_{**}^t.

In summary, our work bounds the number of triangulations in terms of the number of nuclei. Thus, we remain with a new, but hopefully simpler, open question about the problem of exponential growth, namely does the number of face-rooted nuclei with t tetrahedra have an exponential bound in t? While we do not have any mathematical statements about this problem, the methodology of the proof of Theorem 5.17 allows for quite extensive numerical experimentation. The most important insight from this experimentation is as follows: It seems that if T is a nucleus with a k-complicated knot (or even braid), then the addition of (at most) k cones and decomposition with our algorithm leads to a tree of *tetrahedra*. Note that the trefoil knot is 1-complicated. Furthermore, Goodrick [12] showed that the connected sum of k trefoil knots is at least k-complicated.

We have analyzed a certain number of classical examples, with the findings summarized in Table 2.

1.1. The method. The bounds on the number of triangulations are obtained by studying a set of elementary moves, detailed in Sect. 4.1. These moves either decompose the triangulation in two disjoint pieces (by cutting along an interior face with 3 edges on the boundary), or by taking away a tetrahedron with an external face and one internal node. Clearly, this leaves again two 3-balls on which we continue the decomposition. The other operations are "open a ball" along a carefully chosen edge (which we call "split-a-node-along-a-path") or opening one face with 2 external edges. These operations *increase* the number of tetrahedra in the triangulation, but they prepare the moves in which the 3-ball can be cut, and the internal nodes can be eliminated. One of the main novelties of this construction is the observation that this can be done with *few* additional tetrahedra: This follows from a careful analysis of cuts of the 2-dimensional hemisphere attached to any external node. Since this is an important bound, we devote Sect. 3 to its proof. In Sect. 2, we introduce the (standard) terminology for the pieces of any triangulation. In Sect. 4 we combine the 4 moves described above to show how a general triangulation can be decomposed into a set of nuclei. In Sect. 5, we perform the bottom-up procedure and show how one bounds the number of triangulations of the 3-ball in terms of trees whose nodes are (rooted) nuclei, extending in this way the earlier work of [8] and [5].

1.2. Comparison with 2d. It is useful to compare our method to what can be done in 2d. In 2d we have a set of triangles. Any triangulation can be obtained in the following way: First, construct a tree of triangles, adding each triangle with only one face to the existing tree. This object has no internal nodes. Now, glue together adjacent faces of the tree, recursively. In this way one can obtain all triangulations of any polygon.

The inverse operation, while intuitively clear, is a little harder to describe, and we just sketch the procedure. Given any internal node x at distance 1 from the polygon, say, connected to node n of the polygon, we can split the edge (n, x) by doubling the node n into a pair n', n'', so that the edges (n', x) and (x, n'') are now external edges and x is promoted to an external node. All internal nodes can recursively be brought to the surface in this way. We then have a tree, and the tree can be decomposed into triangles by cutting all internal edges with 2 external nodes. At the end, the basic objects are triangles.

Clearly, therefore, the basic objects in 2d are

(2a) internal edges with 2 external nodes,
(2b) internal nodes (at distance 1) from the polygonal boundary.

In 3d, there are many more possibilities, and our procedure will eliminate all those which can be eliminated. The ones which we can deal with are

(3a) internal faces with 3 external edges: this corresponds to case (2a) above and will be cut by cut-a-3-face.
(3b) internal faces with 2 external edges, and therefore one internal edge with 2 external nodes. This resembles (2b) and is dealt with by open-a-2-face.
(3c) an internal node x which is the tip of a tetrahedron t whose opposite face is external. One can just eliminate t and x becomes external. This is the second case which corresponds to (2b). We call this C0 later.
(3d) an internal node x which is the corner of a face f whose opposite edge is external (but not C0). Again, a sub-case of (2b). This is dealt with split-a-node-along-a-path, and will be called C1.

(3e) an internal node x which is the end of an edge e whose opposite end is external (but not C1). Again, a sub-case of (2b). This will be called C2 and reduced to C1 with split-a-node-along-a-path.

The elementary objects are those left over after all these decompositions are performed. In 2d, those objects are just triangles, which makes the counting possible. In 3d these are nuclei. Non-trivial nuclei exist, and they must carry the information about the complications of 3 dimensional topology, since all the other problems have been eliminated. In particular, internal faces of nuclei have 0 or 1 external edges.

2. General Definitions and Notations

2.1. Internal and external objects, flowers. We introduce some notation which we apply to triangulations and tetrahedrizations (which we also call triangulations when no confusion is possible):

We start with triangulations of S^2. These will have $f_{\rm s}$ faces, $n_{\rm s}$ nodes and $e_{\rm s}$ edges, where the subscript s stands for "surface".

We also consider tetrahedrizations of a ball, which are collections of tetrahedra, whose faces, edges and nodes satisfy the usual topological conditions of piecewise linear triangulations. The boundary of such a tetrahedrization is a triangulation of S^2. We say that t is the number of tetrahedra, $f_{\rm tot}$ the number of faces, $e_{\rm tot}$ the number of edges, and $n_{\rm tot}$ the number of nodes. Faces, edges, and nodes which are not among those of the triangulation of S^2 are called *internal*; the others are called external. It will be useful to observe that tetrahedra can have up to 4 external faces, internal faces can have up to 3 external edges, internal edges up to 2 external nodes. We will use the subscript i for internal objects.

Obviously,

$$f_{\rm tot} = f_{\rm s} + f_{\rm i}, \quad e_{\rm tot} = e_{\rm s} + e_{\rm i}, \quad n_{\rm tot} = n_{\rm s} + n_{\rm i}.$$

From the Euler relations and trivial geometry, we have the relations

$$\begin{aligned} t - f_{\rm tot} + e_{\rm tot} - n_{\rm tot} &= -1, \\ f_{\rm s} - e_{\rm s} + n_{\rm s} &= 2, \\ 3f_{\rm s} &= 2e_{\rm s}, \\ 4t &= 2(f_{\rm tot} - f_{\rm s}) + f_{\rm s}. \end{aligned} \tag{2.1}$$

This leaves us with 3 free variables, which we choose as

$$t,\ f_{\rm s}, \text{ and } n_{\rm i}.$$

Note that $f_{\rm s}$ is always even.

Definition 2.1. *We use the term* f-vector *for the three variables* $\langle t, f_{\rm s}, n_{\rm i}\rangle$ *where* $f_{\rm s} \geq 4$.

2.2. Notation and flowers.

- If n_1 and n_2 are 2 distinct nodes, then we denote by (n_1, n_2) the edge connecting the two (if it exists).
- Similarly, if $n_i : i = 1, 2, 3$ are 3 distinct nodes, then (n_1, n_2, n_3) is the face (triangle) with these 3 corners (if it exists).
- If e is an edge and n is a node not in e then (n, e) denotes the face (triangle) with the edge e and the node n (if it exists).

This notation is easily generalized to the case where we consider simplices of dimension 3:

- If n is a node and f is a face not containing n, then (n, f) is the tetrahedron with f as a face and n as the opposite corner (if it exists).
- Similarly, if e is an edge and $n_1, n_2 \notin e$ are 2 distinct nodes then (n_1, n_2, e) is the unique tetrahedron containing all of them (if it exists).
- Finally, if e_1 and e_2 are two edges without common nodes, then (e_1, e_2) is the tetrahedron containing both edges (if it exists).

Paths of nodes connected by edges will be denoted as $\gamma = [n_1, n_2, \ldots, n_k]$ and the union of 2 disjoint paths γ_1, γ_2 (connected by one or both endpoints) will be denoted by $\gamma_1 \cup \gamma_2$.

We next define what we mean by *flowers*. Here, we adapt the conventions to the tetrahedrization of a triangulated sphere S^2. Nodes, edges, and faces are called *external* if they lie entirely in S^2. All others are called *internal*. Consider an external node n_*.[1] We define its 2 flowers:

- The *external flower* $\mathcal{E}(n_*)$ of n_* is the set of all edges e not containing n_* for which (n_*, e) is an external face. Clearly, $\mathcal{E}(n_*)$ is a polygon.
- The *internal hemisphere* $\mathcal{I}(n_*)$ of n_* is the set of all faces f not containing n_* for which (n_*, f) is a tetrahedron. It is easy to see that $\mathcal{I}(n_*)$ is a 2d triangulation whose boundary is the polygon $\mathcal{E}(n_*)$.

We will say that the external flower of an *internal* node x_* is empty. We define the internal (hemi-)sphere $\mathcal{I}(x_*)$ (or simply flower) of x_* as the set of all faces f not containing x_* for which (x_*, f) is a tetrahedron. This is a triangulation of S^2.

We also define the *external flower* $\mathcal{E}(e)$ *of an external edge* e as the 2 nodes n_1, n_2 for which (n_i, e) are 2 external faces. Similarly, the *internal hemisphere* $\mathcal{I}(e)$ *of the external edge* e is defined as the set of all edges e' such that (e, e') is a tetrahedron. By hypothesis, $\mathcal{I}(e)$ is a 1-d triangulation whose boundary is $\mathcal{E}(e)$. Note that there might be internal nodes at distance 1 from e which are not in $\mathcal{I}(e)$.

3. Some Geometrical Considerations: Two-Colored Paths in a Triangulation

Throughout this section all triangulations are 2d triangulations. We describe here properties of paths in a 2d triangulation of a polygon. These properties will play a crucial role when we will bound the effects of moving internal nodes of a 3d triangulation to the surface. However, they are totally independent of the remainder of the paper.

Lemma 3.1. *Let $\mathcal{K}$ be a 2d triangulation with p boundary edges and n internal nodes. Then the number of interior edges in $\mathcal{K}$ is $3n + p - 3$.*

Proof. The proof follows from the Euler relations and is left to the reader.

Lemma 3.2. *Consider a polygon P and let $\mathcal{K}$ be any triangulation whose boundary is P, with $n > 0$ internal nodes. For each node $x \in \mathcal{K} \backslash P$, there are at least 3 simple disjoint paths in the interior of $\mathcal{K}$ connecting it to 3 different points of P.*

Proof. Any triangulation of S^2 is 3-connected. Complete $\mathcal{K}$ into a triangulation of S^2 by adding a cone over its boundary. Let m be the apex of the cone. Then there are at least 3 disjoint simple paths connecting x to m, [7]. Any such path must intersect P, and we take the first intersection point.

[1] We use n_*, m_* and the like for external nodes, and x_*, y_*, ... for internal ones.

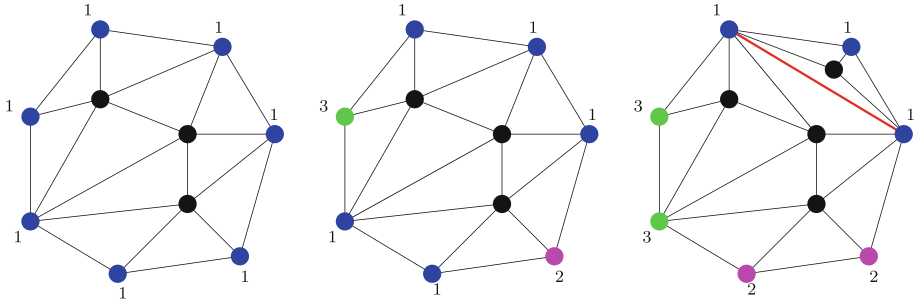

Fig. 1. An illustration of the conditions (K1)–(K3). *Left* Since there is only one label, (K1) is violated. *Center* The region with label 1 is not connected; (K2) is violated. *Right* There is an internal edge (*red*) connecting two nodes with the same label; (K3) is violated (color figure online)

We assume now that the nodes of P are labeled.

Definition 3.3. *A triangulation $\mathcal{K}$ with $P = \partial\mathcal{K}$ is called* admissible *if the following conditions are met:*

K1: The boundary $P = \partial\mathcal{K}$ has at least 2 different labels.
K2: The nodes in $\partial\mathcal{K}$ with a given label form one connected arc of $\partial\mathcal{K}$.
K3: The ends of any internal edge connecting 2 nodes of $\partial\mathcal{K}$ have different labels.

Figure 1 illustrates the definition.

Definition 3.4. *An admissible triangulation of a polygon is called* non-trivial *if it has at least one internal edge. (The only admissible trivial triangulation is an admissible triangle.)*

We begin with an auxiliary lemma.

Lemma 3.5. *Let $\mathcal{K}$ be an admissible triangulation whose boundary is the polygon P. Given two non-adjacent boundary nodes a and b with different labels at least one of the two alternatives below holds:*

(1) *There is a simple path γ joining a and b without any other node in P.*
(2) *There is an edge (x, y) joining the two pieces of $P\setminus\{a, b\}$.*

Postponing the proof of Lemma 3.5 we have

Proposition 3.6. *Assume $\mathcal{K}$ is a non-trivial admissible triangulation with $\partial\mathcal{K} = P$. Then, there exists a simple path γ along internal edges of $\mathcal{K}$ which connects two points in $P = \partial\mathcal{K}$ with* different labels. *It cuts $\mathcal{K}$ in two pieces $\mathcal{K}_{\mathrm{L}}$ and $\mathcal{K}_{\mathrm{R}}$. The path γ can be chosen in such a way that labeling the new boundary piece (namely the interior nodes of γ) in $\mathcal{K}_{\mathrm{L}}$ and $\mathcal{K}_{\mathrm{R}}$ with a label different from the ones used so far, both $\mathcal{K}_{\mathrm{L}}$ and $\mathcal{K}_{\mathrm{R}}$ are again admissible.*

Proof. By admissibility, we know that not all nodes on P have the same label. We distinguish two cases:

– P has more than 3 nodes. In this case, take 2 non-adjacent nodes a and b with different labels and apply Lemma 3.5. If (2) holds, then we take γ as the edge connecting x and y. By (K3) they have different labels. Otherwise, there is an internal path connecting a and b. We take a shortest path, γ.

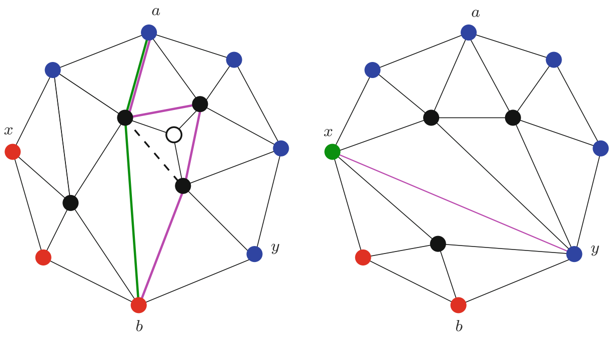

Fig. 2. The 2 alternatives of finding a path connecting two different labels. *Left* There is an interior path between a and b. *Right* There is no such path, but then, one can always find an edge connecting two different labels (by K3), (not necessarily the same as a and b). The *left panel* also illustrates the necessity of choosing a shortest path. For example, choosing the magenta path, the dashed edge will violate (K3) in the next step of the procedure

- P is a triangle. In this case, $\mathcal{K}$ can be seen as a triangulation of the sphere S^2. In particular, it is 3-connected. We take 2 (adjacent) nodes a and b of P with different labels. Since $\mathcal{K}$ is non-trivial and 3-connected, there are (at least) 3 disjoint simple paths connecting a and b. At most 2 of these paths are on the boundary P, which leaves at least 1 path in the interior of the triangulation $\mathcal{K}$. We take a shortest internal path γ connecting a and b.

In all cases, we find 2 boundary nodes with different labels and a shortest internal path γ connecting them. Cutting along the path γ, we obtain 2 pieces $\mathcal{K}_{\mathrm{L}}$ and $\mathcal{K}_{\mathrm{R}}$. If γ is just one edge then inspection shows that (K1)–(K3) hold. If not, (K1) and (K2) are obviously true by construction; we have to show that (K3) is also true. Giving a new label, say L, to the interior nodes of γ, we have to show that there are no edges connecting any two non-consecutive nodes with label L. But if there were, the path would not be minimal.

Proof (Proof of Lemma 3.5). The reader may want to look at Fig. 2. Assume (1) does not hold. This means that one cannot draw 3 disjoint paths between a and b, as the middle one would satisfy (1). We can take the two disjoint paths to go along the two boundary segments between a and b. By Menger's theorem [7], and since a and b are not adjacent, there must then be 2 nodes x and y (other than a or b) such that all paths from a to b must pass through at least one of them. Since the boundary paths are candidates, we see that x and y are in P, one per arc connecting a and b. Consider now the path from a to b along P which goes through x. Modify it so that instead of going through x it goes through the flower of x. We get a new path from a to b which does not go through x. This means that the new path goes through y implying that y is in the flower of x. Thus, x and y are connected by an edge.

This completes the proof.

4. Part I: Reducing any Triangulation into a Set of Nuclei

4.1. The elementary moves. In this section we define the elementary moves which transform any triangulation into a (set of) nuclei. The first two moves, which we call *open-a-2-face* and *cut-a-3-face*, are used to transform any triangulation with no internal nodes

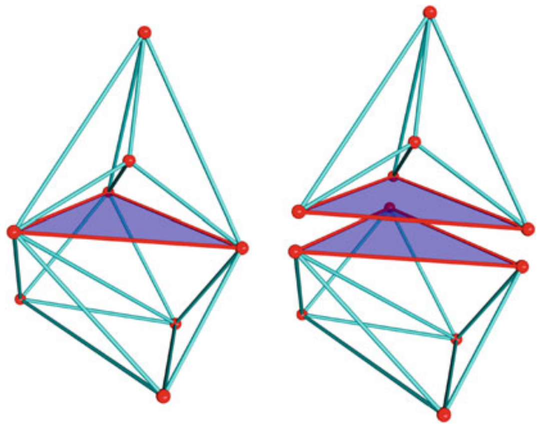

Fig. 3. Sketch of cut-a-3-face

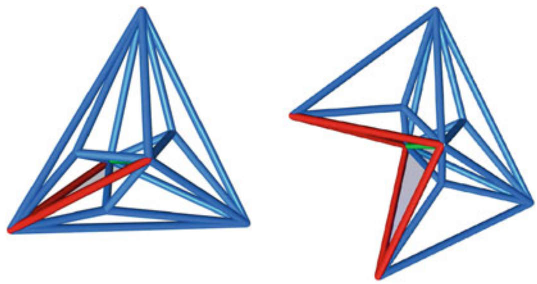

Fig. 4. Sketch of open-a-2-face

into a set of nuclei, and the third and fourth move, which we call *remove-1-tetra* and *split-a-node-along-a-path*, are used to remove all internal nodes of a triangulation.

Henceforth, T will denote a triangulated 3-ball with f-vector $\langle t, f_{\mathrm{s}}, n_{\mathrm{i}}\rangle$.

4.1.1. Cut-a-3-face. Let (n_1, n_2, n_3) be an internal face with its 3 edges on the surface ∂T of T. Then, it cuts the 3-ball T into 2 distinct parts. We simply separate these 2 parts and we get 2 "smaller" 3-balls, as shown in Fig. 3.

If $\langle t, f_{\mathrm{s}}, n_{\mathrm{i}}\rangle$, $\langle t_1, f_{1,\mathrm{s}}, n_{1,\mathrm{i}}\rangle$ and $\langle t_2, f_{2,\mathrm{s}}, n_{2,\mathrm{i}}\rangle$ are the f-vectors of the initial ball and the 2 new ones, then we have

$$t = t_1 + t_2, \quad f_{\mathrm{s}} = f_{1,\mathrm{s}} + f_{2,\mathrm{s}} - 2, \quad n_{\mathrm{i}} = n_{1,\mathrm{i}} + n_{2,\mathrm{i}}.$$

4.1.2. Open-a-2-face. Consider 3 external nodes n_*, n_1, n_2 of T which form a triangle (n_*, n_1, n_2). We assume that (n_*, n_1, n_2) is an internal face, with (n_1, n_2) an internal edge, and the two other edges external. Let $\mathcal{I}$ and $\mathcal{E}$ be the internal and external flower of the external node n_*. As we have already stated, $\mathcal{I}$ is a triangulation with $\partial\mathcal{I} = \mathcal{E}$. By hypothesis, the edge (n_1, n_2) divides $\mathcal{I}$ into 2 distinct sets of faces. The operation open-a-2-face consists in removing n_* and all tetrahedra attached to it, replacing it by $n_{*,1}$ and $n_{*,2}$ and attaching each of these 2 new nodes to all faces of one of the two parts of $\mathcal{I}$, see Fig. 4. *This operation transforms a triangulation of the 3-ball into a*

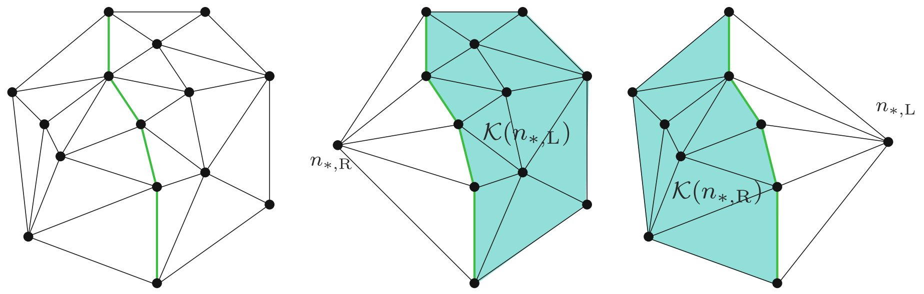

Fig. 5. The *left panel* shows the internal hemisphere $\mathcal{I}(n_*)$ of n_*. The node n_* is not shown, but is connected to every node of $\mathcal{I}(n_*)$. We split n_* into $n_{*,\mathrm{L}}$ and $n_{*,\mathrm{R}}$ along the green path γ. The other 2 panels show the internal hemispheres $\mathcal{I}(n_{*,\mathrm{L}})$ and $\mathcal{I}(n_{*,\mathrm{R}})$ of the 2 new nodes [$n_{*,\mathrm{L}}$ in the *center panel*, $n_{*,\mathrm{R}}$ in the right one (again, they are not shown but connected to every node we draw)]. Notice that each internal node of the green path γ is at distance 1 from $n_{*,\mathrm{R}}$ resp. $n_{*,\mathrm{L}}$. Also, the edges originating in $n_{*,s}$, $s \in \{\mathrm{L}, \mathrm{R}\}$ have been added during the split

triangulation of the 3-ball. If $\langle t, f_\mathrm{s}, n_\mathrm{i}\rangle$ and $\langle t', f'_\mathrm{s}, n'_\mathrm{i}\rangle$ are the f-vectors of the initial ball and the resulting ball, then we have

$$t' = t, \quad f'_\mathrm{s} = f_\mathrm{s} + 2, \quad n'_\mathrm{i} = n_\mathrm{i}.$$

We will say that the f-vector changes by $\langle 0, +2, 0\rangle$.

4.1.3. Remove-1-tetra.

Definition 4.1. *A* removable tetrahedron *is any tetrahedron t with one internal node and one external face.*

The operation remove-1-tetra is as follows: let $t_* = (x_*, n_1, n_2, n_3)$ be a removable tetrahedron with internal node x_* and external face (n_1, n_2, n_3). We simply remove t_* and its external face; the internal node x_*, the 3 internal edges and the 3 internal faces of t_* all become external. The f-vector $\langle t, f_\mathrm{s}, n_\mathrm{i}\rangle$ changes to $\langle t - 1, f_\mathrm{s} + 2, n_\mathrm{i} - 1\rangle$; the change of f-vector is $\langle -1, 2, -1\rangle$.

4.1.4. Split-a-node-along-a-path, hemispheres and pieces. Consider an external node n_* of T and its internal hemisphere $\mathcal{I} = \mathcal{I}(n_*)$, see Fig. 5 for an illustration. By definition of a triangulation, $\mathcal{I}$ is a 2d triangulation of a polygon.

Definition 4.2. *A* splitting path γ *is any simple path in $\mathcal{I}$ which connects two different vertices on $\partial\mathcal{I}$ and contains no edge of $\partial\mathcal{I}$.*

Let γ be a splitting path. Clearly it divides $\mathcal{I}$ into 2 pieces $\mathcal{K}_\mathrm{L}$ and $\mathcal{K}_\mathrm{R}$ with $\mathcal{I} = \mathcal{K}_\mathrm{L} \cup \mathcal{K}_\mathrm{R}$ and $\mathcal{K}_\mathrm{L} \cap \mathcal{K}_\mathrm{R} = \gamma$.

The move split-a-node-along-a-path γ is defined as follows:

1. Remove the node n_* and all tetrahedra having n_* as a corner.
2. Add 2 new nodes $n_{*,\mathrm{L}}$ and $n_{*,\mathrm{R}}$.
3. For each face $f_* \in \mathcal{K}_\mathrm{L}$ add the tetrahedron $(n_{*,\mathrm{L}}, f_*)$.
4. For each face $f_* \in \mathcal{K}_\mathrm{R}$ add the tetrahedron $(n_{*,\mathrm{R}}, f_*)$.
5. For each edge $e \in \gamma$ add the tetrahedron $(n_{*,\mathrm{L}}, n_{*,\mathrm{R}}, e)$.

Note that by construction, one of the nodes on $\partial\mathcal{K}_\mathrm{L}$ is $n_{*,\mathrm{R}}$, and the edges in $\mathcal{K}_\mathrm{L}$ originating from $n_{*,\mathrm{R}}$ reach (the image of) γ. Analogous statements hold for $\mathcal{K}_\mathrm{R}$.

Definition 4.3. *In the construction above, we refer to* $\mathcal{K}(n_{*,\mathrm{L}}) = \mathcal{K}_\mathrm{L}$ *as the left* piece. *It is simply the subtriangulation obtained from the hemisphere* $\mathcal{I}(n_{*,\mathrm{L}})$ *after removing the cone connecting* $n_{*,\mathrm{R}}$ *to every node of* γ. *Similarly, we define the right piece* $\mathcal{K}(n_{*,\mathrm{R}}) = \mathcal{K}_\mathrm{R}$.

Remark 4.4. Hemispheres $\mathcal{I}$ and pieces $\mathcal{K}$ will play an important role in our construction. Some statements will be given for hemispheres, others for pieces and so it is important to be able to distinguish between the two definitions.

Remark 4.5. A splitting path γ is always associated with a hemisphere $\mathcal{I}$ and not with a piece $\mathcal{K}$. We will see that, under some conditions, a simple path $\tilde{\gamma}$ connecting two nodes of the boundary of a piece $\mathcal{K}$ can be extended into a splitting path γ.

Lemma 4.6. *The move split-a-node-along-a-path transforms a 3-ball into a 3-ball. The f-vector* $\langle t, f_\mathrm{s}, n_\mathrm{i}\rangle$ *is mapped to* $\langle t + |\gamma|, f_\mathrm{s} + 2, n_\mathrm{i}\rangle$, *where* $|\gamma|$ *is the number of edges in* γ.

The f-vector changes by $\langle |\gamma|, +2, 0\rangle$. In particular, the number of tetrahedra *increases*. But we will show that this increase can be controlled.

Proof. The count of the f-vector is as follows: Removing and adding the tetrahedra in steps 1, 3, 4 above does not change their number. The number of external faces increases by two, namely the two external faces sharing the new edge $(n_{*,\mathrm{L}}, n_{*,\mathrm{R}})$. And each internal face (n_*, e) which connected n_* to an edge e in γ gives rise to a new tetrahedron $(n_{*,\mathrm{R}}, n_{*,\mathrm{L}}, e)$. There are $|\gamma|$ such faces and so the f-vector is seen to change by $\langle |\gamma|, +2, 0\rangle$, as asserted.

4.2. Summary. In the sequel, we want to bound the effect of removing internal nodes, since our building blocks are the nuclei, which do not have any internal nodes. Eliminating the internal nodes will cost the addition of tetrahedra, and the issue here is how many are needed to obtain a ball without internal nodes. Internal nodes disappear when we perform the remove-1-tetra operation, and only then.

Before starting the bounds proper, we explain here the point of our construction, based on the evolution of the f-vectors $\langle t, f_\mathrm{s}, n_\mathrm{i}\rangle$. Open-a-2-face costs a change $\langle 0, 2, 0\rangle$, and split-a-node-along-a-path costs $\langle |\gamma|, 2, 0\rangle$, where $|\gamma|$ is the length of the path along which we cut. In principle, each path γ might have a length proportional to the number of nodes, which in turn would imply that the sum of the lengths of all paths exceeds $\mathcal{O}(n_\mathrm{tot}^2)$. So one needs a strategy which improves this naive bound.

While we cut, new external edges appear, and also, new external edges appear when we remove a tetrahedron which costs $\langle -1, 2, -1\rangle$. But it is only this operation which reduces the number of internal nodes. So, there are two opposing tendencies. One is the preparation of promoting an internal node into an external one, and it *adds* many tetrahedra, and the other is remove-1-tetra, which reduces the number of internal nodes by 1.

The real issue is thus to bound the *number of added tetrahedra per removed internal node*. We will perform this bound in terms of the number e_i of internal edges. Our main result is Corollary 4.13 which says that the number of internal edges grows by no more than $C_\Delta(t + n_\mathrm{i})$. The Euler relations (2.1) allow to express t as a function of e_s, f_s, and n_i,

$$t = e_\mathrm{i} - n_\mathrm{i} + f_\mathrm{s}/2 - 1. \tag{4.1}$$

Therefore, and since $n_\mathrm{i} < 4t$, $f_\mathrm{s} = 2n_\mathrm{s} - 4 < 4e_\mathrm{i}$ and $e_\mathrm{i} < 6t$, Corollary 4.13 implies that the elimination of all n_i internal nodes leads to an f-vector of the form

$$\langle t, f_\mathrm{s}, n_\mathrm{i}\rangle \to \langle t', f'_\mathrm{s}, 0\rangle,$$

with

$$e'_\mathrm{i} < e_\mathrm{i} + 5C_\Delta \cdot t < (6 + 5C_\Delta) \cdot t = C/4 \cdot t,$$

and therefore

$$f'_\mathrm{s} < C \cdot t, \quad t' < C \cdot t,$$

with a finite constant C which is *independent* of the triangulation.

4.3. Removing internal nodes. This is the most difficult, and novel, part of our construction.

4.3.1. Definitions and strategy. Given any triangulation, the *depth* D_x of a node x is the minimal number of connected edges needed to reach the boundary, starting from x. We also say that the triangulation has maximal depth $d_{\max} = \max_{n\in T} D_n$. Our strategy consists in reducing the depth of all internal nodes by 1. The depth will be reduced by working on all internal nodes of depth 1 and moving them to the surface. If the maximal depth of a triangulation was $d_{\max}$ it will end up being of depth $d_{\max} - 1$. We repeat this procedure until no internal nodes remain.

So, consider now an internal node x_* of depth 1. It comes in 3 flavors which we call C0–C2:

C0: x_* is the internal node of a removable tetrahedron.
C1: x_* is not of type C0 but is in a face (x_*, n_*, m_*) where (n_*, m_*) is an external edge.
C2: x_* is neither of type C0 nor C1.

Obviously, in case C0, we would just remove the tetrahedron, promoting x_* to the surface. The other cases are more complicated and need a careful estimate in terms of edges and faces which appear when the nodes are brought to the surface. No node will ever disappear in these constructions, but some will be doubled (split) and edges will be added.

So we begin with a triangulation T_0 of maximal depth $d_{\max} > 0$. We define $\mathcal{L}_\ell = \{n : D_n = \ell\}$, the set of nodes at depth ℓ in the triangulation T_0. These (original) depths should be viewed as labels assigned to each node. If a node is split, its children inherit the label. If a node comes closer to the surface in the procedure below, its (actual) depth decreases, but its label (the original depth) does *not* change.

We then iterate the following 4 steps until no internal nodes remain, for $\ell = 0, \dots d_{\max} - 1$. Each iteration transforms the ball T_ℓ into a new ball $T_{\ell+1}$, such that the nodes of $\mathcal{L}_{\ell+1}$ are external.

Assume iteration $\ell - 1$ is completed: Then we say that in T_ℓ the nodes of $\mathcal{L}_\ell$ have become external. Given such a node $n_* \in \mathcal{L}_\ell$, we consider its hemisphere $\mathcal{I}(n_*)$ (in T_ℓ).

An internal node $x_* \in \mathcal{I}(n_*)$ of type C2 can be promoted to an internal node of type C1 by drawing a path $\gamma \subset \mathcal{I}(n_*)$ that goes through it and splitting n_* into $n_{*,\mathrm{L}}$ and $n_{*,\mathrm{R}}$ along γ. Then, $n_{*,\mathrm{R}}$ is among the edges $\mathcal{E}(n_{*,\mathrm{L}})$ and therefore $(n_{*,\mathrm{L}}, n_{*,\mathrm{R}}, x_*)$ is a face with the external edge $(n_{*,\mathrm{L}}, n_{*,\mathrm{R}})$, so that x_* is now of type C1.

In a similar manner, a node $x_* \in \mathcal{I}(n_*)$ of type C1 can be promoted into an internal node of type C0 by drawing a path $\gamma \subset \mathcal{I}(n_*)$ which contains the edge (x_*, y). Here, y is the external node of the face (x_*, n_*, y) which defined x_* as a node of type C1. Splitting n_* along γ, the tetrahedron $(n_{*,\mathrm{L}}, n_{*,\mathrm{R}}, y, x_*)$ becomes removable.

Finally, any internal node of type C0 can be made external by simply removing one tetrahedron.

Thus, to move all nodes of depth 1 to the surface we proceed in 4 steps (3 sweeps).

- **Step 1 (Sweep C2→C1)**: We promote all the x_* of type C2 in the following order: For each $n_* \in \mathcal{L}_\ell$, we promote all internal nodes of $\mathcal{I}(n_*)$ of type C2 into internal nodes of type C1. We will show that this can be done in such a way that every internal edge of the triangulation $\mathcal{I}(n_*)$ belongs to at most 1 of the splitting paths (as defined in Sect. 4.1.4).
  When this first step is complete, all internal nodes initially at depth 1 will be of type C1 or C0. There appears a new set $\mathcal{M}_\ell$ of external nodes containing the nodes of $\mathcal{L}_\ell$ which were not split and new external nodes obtained by the splitting.
- **Step 2 (Sweep C1→C0)**: We promote all the x_* of type C1 in the following order: For each $n_* \in \mathcal{M}_\ell$, we promote all promotable internal nodes of $\mathcal{I}(n_*)$ of type C1 into internal nodes of type C0.[2] We will show that this can be done in such a way that every internal edge of the triangulation $\mathcal{I}(n_*)$ belongs to at most 1 of the splitting paths (as defined in Sect. 4.1.4).
- **Step 3 (Sweep C0→external)**: Finally, we make each node of type C0 external by removing one tetrahedron.
- **Step 4**: At this point every internal node has been moved up one level of depth. We have now a new triangulation $T_{\ell+1}$ of the ball, of maximal depth $d_{\max} - \ell - 1$.

Remark 4.7. The reader should be aware that the 4 steps are repeated until no internal nodes remain. However, these steps are not independent, and the proof of our bound does depend on the precise definition of $\mathcal{L}_\ell$.

Since $d_{\max} < \infty$, the procedure will end after a finite number of iterations of C2→C1→C0. We number these steps from $\ell = 0$ to $\ell = d_{\max} - 1$.

4.3.2. Reducing C2-nodes to C1-nodes. Given an external node $n_* \in \mathcal{L}_\ell$, we now describe in detail the algorithm which promotes the internal nodes of type C2 in $\mathcal{I} = \mathcal{I}(n_*)$ to type C1. This is achieved by a succession of carefully chosen moves of type split-a-node-along-a-path.

Each of these cuts produces a "left" and a "right" piece, which are then cut again into left and right pieces, until only triangles remain. The pieces will be called $\mathcal{K}_{\mathbf{S}} = \mathcal{K}(n_{*,\mathbf{S}})$, where **S** is a sequence of letters L and R which designate the successive choices of left and right. They are all admissible.

Thus, we construct a binary tree of pieces (see Fig. 6). In detail:

1. Label the nodes of $\partial\mathcal{I}$ from -1 to $-|\partial\mathcal{I}|$.
2. The hemisphere $\mathcal{I}$ is an admissible triangulation in the sense of Definition 3.3. Proposition 3.6 implies the existence of a shortest path γ which connects two nodes of $\partial\mathcal{I}$ (with different labels). We choose this path γ.
3. After splitting along this path, $\mathcal{I}$ is divided in two pieces, as shown in Fig. 5. The two pieces are called $\mathcal{K}_\mathrm{L}$ and $\mathcal{K}_\mathrm{R}$. The splitting has replaced n_* by $n_{*,\mathrm{L}}$ and $n_{*,\mathrm{R}}$ and

[2] A node x_* of type C1 can be promoted to C0 only if it is connected to a node of $\mathcal{E}(n_*)$. This might not be true for all n_* for which $x_* \in \mathcal{I}(n_*)$ but for every x_* there are at least two n_* for which it is promotable.

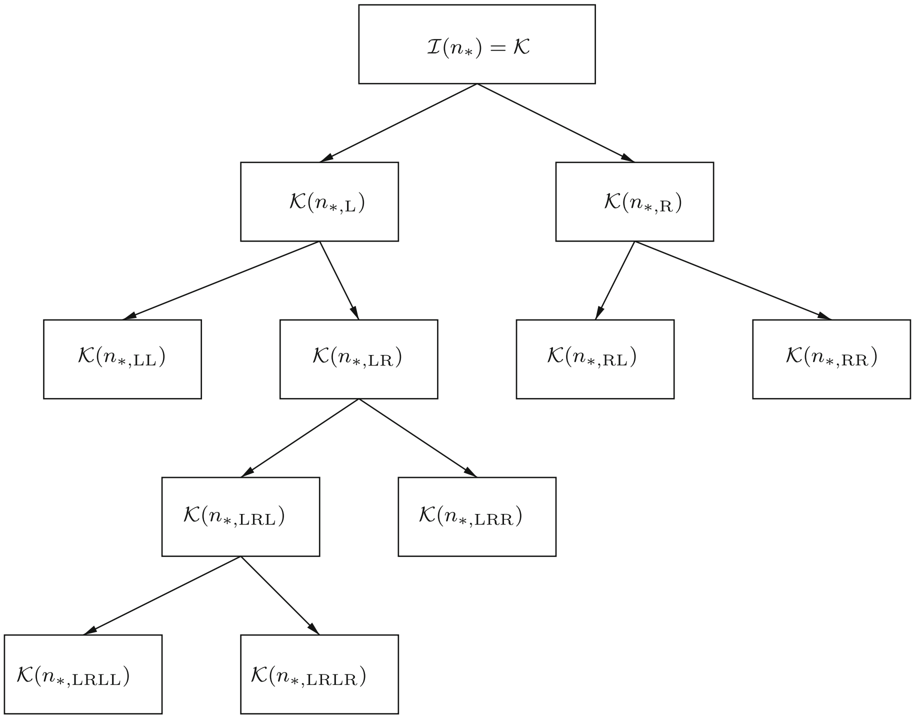

Fig. 6. An example of a binary tree of pieces associated with the hemisphere of an external node n_* containing 6 triangles

$\mathcal{I}(n_{*,\mathrm{L}})$ is actually just $\mathcal{K}_\mathrm{L}$ with the cone between $n_{*,\mathrm{R}}$ and γ added. This also means that $n_{*,\mathrm{R}}$ is in the external flower $\mathcal{E}(n_{*,\mathrm{L}})$ of $n_{*,\mathrm{L}}$. Analogous terminology is used for the other half. At this point, $\mathbf{S}$ is equal to L or R, and we continue with $\mathbf{S} = \mathrm{L}$ (and do later $\mathbf{S} = \mathrm{R}$).

4. If $\mathcal{K}_\mathbf{S}$ is a triangle, we are done (for this branch of the tree).
5. Label all nodes on $\partial\mathcal{K}_\mathbf{S}$ which had no label with the label $\hat{\mathbf{S}}$, where $\hat{\mathbf{S}}$ is obtained from $\mathbf{S}$ by exchanging the last letter, cf. Fig. 7. In this way, the newly labeled nodes are connected to $n_{*,\hat{\mathbf{S}}}$ in $\mathcal{I}(n_{*,\mathbf{S}})$.
6. Considering $\mathcal{K}_\mathbf{S}$, Proposition 3.6 implies the existence of a new shortest path $\tilde{\gamma}_\mathbf{S}$ which connects two nodes of $\partial\mathcal{K}_\mathbf{S}$ with different labels.
7. We extend the path $\tilde{\gamma}_\mathbf{S}$ as follows:
   - If the end of $\tilde{\gamma}_\mathbf{S}$ has a negative label, we do nothing.
   - If the label of the end of $\tilde{\gamma}_\mathbf{S}$ is some sequence $\mathbf{S}'$ and if the node $n_{*,\mathbf{S}'}$ still exists, *i.e.*, has not been split yet, we connect the end to $n_{*,\mathbf{S}'}$ by one edge.
   - If the label of the end of $\tilde{\gamma}_\mathbf{S}$ is some sequence $\mathbf{S}'$ and if the node $n_{*,\mathbf{S}'}$ was previously split, then at least 1 of its children has replaced it in the flower of $n_{*,\mathbf{S}}$ (see Fig. 8). In this case, we connect the end to any one of the corresponding children.

   Doing this for both ends we obtain a path $\gamma_\mathbf{S}$.
8. Perform a split-a-node-along-a-path on $\gamma_\mathbf{S}$ and continue with step 4 for the pieces $\mathcal{K}_{\mathbf{S}\mathrm{L}}$ and $\mathcal{K}_{\mathbf{S}\mathrm{R}}$.

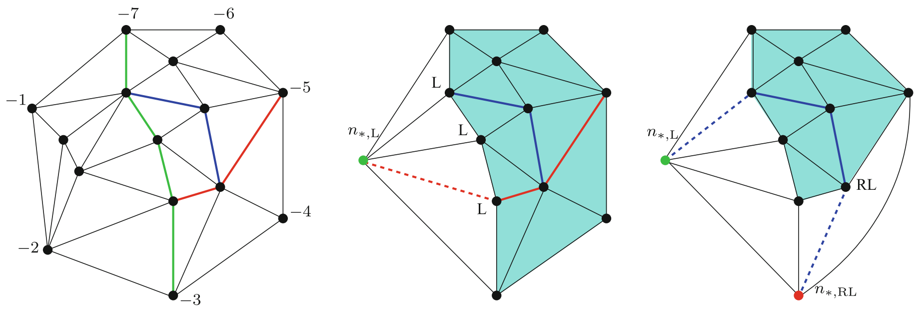

Fig. 7. The *left panel* shows the internal flower $\mathcal{I}(n_*)$ of n_*. We split it in succession along the *green*, *red* and *blue* paths. We first split n_* into $n_{*,\mathrm{R}}$ and $n_{*,\mathrm{L}}$ along the *green* path. The *middle panel* shows $\mathcal{I}(n_{*,\mathrm{R}})$ and the *green* node is $n_{*,\mathrm{L}}$. The *shaded region* is $\mathcal{K}_\mathrm{R}$ and the new labels are L. One end of the red path has a label which is a negative integer. while the other has the label L and must therefore be connected to $n_{*,\mathrm{L}}$. We obtain the cutting path γ_R, and after the cut, we obtain two pieces $\mathcal{K}_{\mathrm{RR}}$ and $\mathcal{K}_{\mathrm{RL}}$. In the third panel we show the hemisphere of $n_{*,\mathrm{RR}}$. The *blue* path has labels L and RL at its extremities, which must therefore be connected to $n_{*,\mathrm{L}}$ and $n_{*,\mathrm{RL}}$. This defines the cutting path γ_{RR}. Note that it always suffices to add at most 2 dashed segments

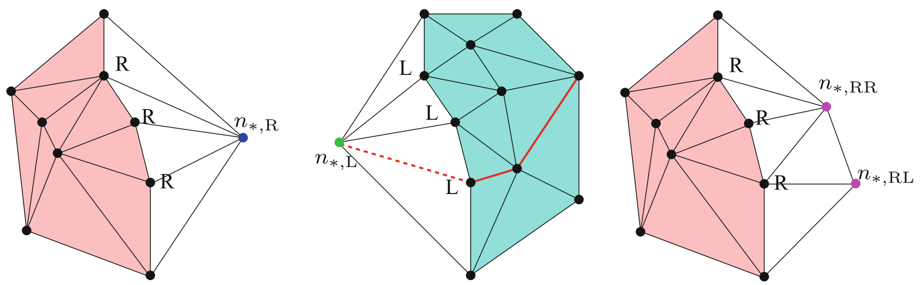

Fig. 8. The *first two panels* show the hemispheres of $n_{*,\mathrm{L}}$ and $n_{*,\mathrm{R}}$ obtained by splitting the node n_* of Fig. 7 along the *green* path. Note that $n_{*,\mathrm{R}}$ is in $\mathcal{E}(n_{*,\mathrm{L}})$ and $n_{*,\mathrm{L}}$ is in $\mathcal{E}(n_{*,\mathrm{R}})$. The *third panel* shows what happens to $\mathcal{I}(n_{*,\mathrm{L}})$ if we split $n_{*,\mathrm{R}}$ along the *red path* of the *second panel* into $n_{*,\mathrm{RR}}$ and $n_{*,\mathrm{RL}}$. Note that at least 1 child of $n_{*,\mathrm{R}}$ (in this case both of them) is still in $\mathcal{E}(n_{*,\mathrm{L}})$

Remark 4.8. The boundary of a hemisphere $\mathcal{I}_\mathbf{S} = \mathcal{I}(n_{*,\mathbf{S}})$ is composed of 2 types of nodes:

1. Nodes with a negative label which are part of the original boundary $\mathcal{E}$.
2. The children $n_{*,\mathbf{S}'}$ of the original node n_*, where $\mathbf{S}'$ is a sequence of R's and L's.

The boundary of a piece $\mathcal{K}_\mathbf{S}$, which is a sub-triangulation of $\mathcal{I}_\mathbf{S}$, is also composed of two types of nodes:

1. Nodes with a negative label which are part of the original boundary $\mathcal{E}$, and therefore they are part of the boundary of the hemisphere $\mathcal{I}_\mathbf{S}$ as well.
2. The other nodes whose label is some sequence S'. These nodes satisfy the following two conditions:
   (a) All nodes of $\partial\mathcal{K}_\mathbf{S}$ with the same label form a connected arc of $\partial\mathcal{K}_\mathbf{S}$.
   (b) If a node $y \in \partial\mathcal{K}_\mathbf{S}$ has the label $\mathbf{S}'$, then y is an internal node of the hemisphere $\mathcal{I}_\mathbf{S}$ seen as a 2d triangulation. Furthermore, the node $n_{*,\mathbf{S}'}$ (or at least 1 of its children if $n_{*,\mathbf{S}'}$ was previously split, (see Fig. 8)) is in $\partial\mathcal{I}_\mathbf{S}$ and the edge connecting y and $n_{*,\mathbf{S}'}$ (or its corresponding child) is an internal edge of the triangulation $\mathcal{I}_\mathbf{S}$.

Theorem 4.9. *The algorithm promotes all of the internal nodes of $\mathcal{I}$ of type C2* (*and depth* 1) *into nodes of type C1. Furthermore, every edge in $\mathcal{I}\backslash\partial\mathcal{I}$ is in at most one path. No new nodes of type C2 at depth 1 are created.*

Proof. We first check that the different steps of the algorithm can be performed. Steps 1–3 follow from the definition of split-a-node-along-a-path. Steps 4 and 5 need no verification. Step 6 relies on Proposition 3.6, which implies the existence of a (shortest) path $\tilde{\gamma}_{\mathbf{S}}$, cutting the admissible piece $\mathcal{K}_{\mathbf{S}}$ into two admissible pieces $\mathcal{K}_{\mathbf{SL}}$ and $\mathcal{K}_{\mathbf{SR}}$.

In step 7, we need to make sure that the path $\gamma_{\mathbf{S}}$ connects two *different* nodes of $\mathcal{E}(n_{*,\mathbf{S}})$ which is also $\partial\mathcal{I}(n_{*,\mathbf{S}})$, to be distinguished from $\partial\mathcal{K}(n_{*,\mathbf{S}})$. The whole construction of labels has been done with this aim in mind. Note that if a node u has a negative label, we do nothing because any node u with a negative label is part of the original boundary $\mathcal{E}(n_*)$, implying that if $u \in \mathcal{I}(n_{*,\mathbf{S}})$, then $u \in \mathcal{E}(n_{*,\mathbf{S}})$ for any child $n_{*,\mathbf{S}}$ of n_*. On the other hand, if the label is the sequence $\mathbf{S}'$, then by construction (step 5), u is connected to $n_{*,\mathbf{S}'}$ (or to one of its children) with one edge. Since the labels are different by construction, the path γ is a splitting path, and therefore a cut along it is possible. In step 8, we need to verify that the cut can indeed be done, and that the algorithm can be applied to the children of the $\mathcal{K}$ which was just cut. But this is the content of Proposition 3.6, which shows that the cut can be done in such a way that the children are admissible in the sense of Definition 3.3.

Since new paths are always constructed in the interior of $\mathcal{K}$, and the $\mathcal{K}$'s are cut along them, it is obvious that no edge (of the original hemisphere $\mathcal{I}(n_*)$) is covered by more than one path.

Finally, to finish the proof, we note that every node of $\mathcal{I}\backslash\partial\mathcal{I}$ belongs to at least 1 path $\tilde{\gamma}_{\mathbf{S}}$, since the only pieces remaining at the end of the algorithm are triangles. In particular, this implies that every internal node of type C2 in $\mathcal{I}$ is promoted to a node of type C1.

4.3.3. Reducing C1-nodes to C0-nodes. Let T be a triangulation of a ball. Consider an external node n_* of T and let $\mathcal{I} = \mathcal{I}(n_*)$ be its internal hemisphere. Furthermore, assume that all nodes of $\mathcal{I}$ are either external (with regard to T) or internal of type C0 or C1 but not C2. We now describe the algorithm which promotes all the internal nodes of type C1 of $\mathcal{I}$ to internal nodes of type C0. The approach is somewhat different from that of the previous section. Indeed promoting an internal node x of type C2 to an internal node of type C1 is done by splitting some external node n_* along a path going through x. However, let $x \in \mathcal{I}(n_*)$ be an internal node of type C1 and let (x, y, n_*) be an internal face which defines x as C1; by hypothesis, $y \in \partial\mathcal{I}$. Promoting x to an internal node of type C0 is done by splitting n_* along a path which contains the edge (y, x). So we have to make sure such a path exists.

For every internal node x of type C1 in $\mathcal{I}(n_*)$ we choose one of the $y \in \partial\mathcal{I}$ for which (x, y, n_*) is an internal face and call it $y(x)$. We define

$$\mathcal{Y} = \{(x, y(x)) \mid x \text{ is C1}\}.$$

We will eliminate elements in the list $\mathcal{Y}$ by iterating an algorithm similar to the one in the previous section, until none are left. A binary tree of left and right pieces will be formed in the process (see Fig. 6).

At the first step of this algorithm, this tree only contains one element, namely the hemisphere $\mathcal{I}$. We will form a tree of $\mathcal{K}$'s as before, starting at $\mathcal{K} = \mathcal{I}$.

The algorithm starts with steps 1 and 2 below, and then repeats the other steps until it stops.

1. Pick an edge $(x, y) = (x, y(x)) \in \mathcal{Y}$.
2. By hypothesis, $y \in \partial\mathcal{I}(n_*)$. By Lemma 3.2 there is a second, disjoint, simple path connecting x to a node $z \in \partial\mathcal{I}(n_*)$, $z \neq y$. This defines a splitting path γ connecting 2 distinct nodes y and z of $\partial\mathcal{I}(n_*)$. Similarly to the previous section, we split n_* along γ into $n_{*,\mathrm{R}}$ and $n_{*,\mathrm{L}}$. We add the 2 new pieces $\mathcal{K}(n_{*,\mathrm{R}})$ and $\mathcal{K}(n_{*,\mathrm{L}})$ as two leaves of $\mathcal{K}$ in the tree. We remove the edge (x, y) from the list $\mathcal{Y}$. Note that the path γ might promote a second internal node x' of type C1 into a node of type C0, if the edge (x', z) is in the list $\mathcal{Y}$ and in the path γ. In that case, both edges (x, y) and (x', z) are removed from $\mathcal{Y}$.
3. If the list $\mathcal{Y}$ is empty, we are done.
4. Pick an edge $(x, y) \in \mathcal{Y}$.
5. Find the piece $\mathcal{K}(n_{*,s_1,\dots,s_k})$, where $s_i \in \{\mathrm{L}, \mathrm{R}\}$, among the leaves of the binary tree which contains the edge (x, y). We use the abbreviations $\mathbf{S} = s_1, \dots, s_k$ and $n_{*,\mathbf{S}}$. The edge (x, y) belongs to exactly one piece.[3]
6. Observe that the node y is in $\partial\mathcal{I}(n_{*,\mathbf{S}}) \cap \partial\mathcal{I}(n_*)$.[4] The edge (x, y) gives us the first simple path connecting x to $\partial\mathcal{I}(n_{*,\mathbf{S}})$ since by construction, it is an internal edge of $\mathcal{K}(n_{*,s_1,\dots,s_k})$. We still need to find the other part of the splitting path $\gamma_{\mathbf{S}}$:
   • If x is in the interior of $\mathcal{K}(n_{*,\mathbf{S}})$, by Lemma 3.2 there is a second independent path connecting x to a node $z \in \partial\mathcal{K}(n_{*,\mathbf{S}})$, $z \neq y$. If z is also in $\partial\mathcal{I}(n_{*,\mathbf{S}})$ we have found a $\gamma_{\mathbf{S}}$ along which we can cut. Note that in this case, the path $\gamma_{\mathbf{S}}$ might promote a second node x' of type C1; this happens if $z \in \partial\mathcal{I}(n_*)$ and (x', z) is an edge of $\gamma_{\mathbf{S}}$. If $z \notin \partial\mathcal{I}(n_{*,\mathbf{S}})$, the path $\gamma_{\mathbf{S}}$ is obtained by adding the edge which connects z to the tip of the cone.[5]
   • If x is not in the interior of $\mathcal{K}(n_{*,\mathbf{S}})$, $\gamma_{\mathbf{S}}$ is found by connecting x to a tip of one of the cones attached to $\mathcal{K}(n_{*,\mathbf{S}})$[6] (see Footnote 5).
7. We split $n_{*,\mathbf{S}}$ along the path $\gamma_{\mathbf{S}}$ and add the 2 new pieces $\mathcal{K}(n_{*,\mathbf{S}\mathrm{R}})$ and $\mathcal{K}(n_{*,\mathbf{S}\mathrm{L}})$ to the tree as leaves of $\mathcal{K}(n_{*,\mathbf{S}})$. Note that $\mathcal{K}(n_{*,\mathbf{S}})$ is no longer a leaf of the tree and will never be encountered in the remaining steps of the algorithm. Finally, we remove the edge (x, y) (and eventually (x', z) if x' is also promoted by $\gamma_{\mathbf{S}}$) from the list $\mathcal{Y}$.
8. We continue with step 3.

The algorithm stops when all internal nodes of type C1 of $\mathcal{I}(n_*)$ have been promoted to C0. Since each branch of the tree is used at most once and since we never cut along the boundary of any $\mathcal{K}_{\mathbf{S}}$ we have shown:

Theorem 4.10. *The algorithm promotes all of the internal nodes of $\mathcal{I}$ of type C1* (*and depth* 1) *into nodes of type C0. Furthermore, every edge in $\mathcal{I}\backslash\partial\mathcal{I}$ is in at most one path, and no new nodes of type C1 or C2 are created.*

4.3.4. Change of the f-vector by removing all internal nodes. We now bound the change of the f-vector which results from transforming a triangulation of maximal depth $d_{\max}$ to one with maximal depth $d_{\max} - 1$, until no internal nodes remain.

Unfortunately, the f-vector alone is not good enough for efficient bounds, since new edges will appear in the construction, and we need to keep track not only of the total

[3] Note that the only edges which are common to more than one piece are the edges of the paths along which we already cut. Since (x, y) is still in the list $\mathcal{Y}$, it cannot be such an edge.

[4] By hypothesis, $y \in \partial\mathcal{I}(n_*)$ and therefore also $y \in \partial\mathcal{I}(n_{*,\mathbf{S}}) \cap \partial\mathcal{I}(n_*)$.

[5] The distance between $\partial\mathcal{I}(n_{*,\mathbf{S}})$ and any node in $\partial\mathcal{K}(n_{*,\mathbf{S}})$ is at most 1, see Fig. 7. The node z belongs to a path $\gamma_{\mathbf{S}'}$ along which we already cut. This implies that z is connected to $n_{*,\mathbf{S}'\mathrm{L}}$ or $n_{*,\mathbf{S}'\mathrm{R}}$, called the *tip of the cone* associated with z.

[6] Note that the node y is *not* on a tip of a cone but is on the original boundary $\partial\mathcal{I}(n_*)$, guaranteeing that $\gamma_{\mathbf{S}}$ is not a closed loop.

number of internal edges as the procedure continues, but also how many there are on each (current) depth. By definition an edge is either on one depth or connects two adjacent depths, and a face also connects at most 2 depths.

Given the original triangulation T, the bookkeeping will be done by associating with each node x_* its *original depth* $d(x_*)$.

As we are going to split nodes, we also define $d(x_{*,\mathrm{R}}) = d(x_{*,\mathrm{L}}) = d(x_*)$, and similarly for all further splittings.

Definition 4.11. *For each* $d : 0 \le d \le d_{\max}$ *we set:*

- a_d *as the number of internal edges* (x, y) *with* $d(x) = d$ *and* $d(y) = D(y) = d + 1$.
- b_d *as the number of internal edges* (x, y) *with* $d(x) = d$ *and* $d(y) = d$.
- f'_d *is the number of internal faces* (x, y, z) *with* $d(x) = d$ *and* $d(y) = d + 1$ *(this implies* $d(z) \in \{d, d + 1\}$*).*

We also say that $a_{-1} = a_{d_{\max}} = 0$ and $f'_{-1} = f'_{d_{\max}} = 0$. Note that all these constants will not change during the iterations since they are counters of the initial triangulation.

As every node is connected to nodes of the same depth or to depths differing by at most 1, the following obvious relations hold:

$$\sum_d (a_d + b_d) = e, \quad \sum_d f'_d \le f_\mathrm{i}, \tag{4.2}$$

where e is the number of internal edges, and f_i is the number of internal faces.

Moving nodes to the surface causes the creation of new edges and faces. Our study is based on a careful bound of this growth in terms of the counters of the initial triangulation introduced in Definition 4.11. Let Δ_ℓ denote the increase of the number of internal edges obtained by performing the steps C2→C1→C0→external at iteration ℓ.

Proposition 4.12. *There is a constant* C' *such that*

$$\begin{aligned} \Delta_\ell &\le C' \,(a_\ell + a_{\ell-1} + b_\ell + f'_\ell), \quad \textit{for } \ell > 0, \\ \Delta_0 &\le C' \,(a_0 + b_0 + n_\mathrm{s}). \end{aligned} \tag{4.3}$$

Corollary 4.13. *Eliminating all internal nodes of a triangulation* T *with f-vector* $\langle t, f_\mathrm{s}, n_\mathrm{i}\rangle$ *leads to a total increase* Δ *of internal edges which is bounded by*

$$\Delta \le C \,(t + n_\mathrm{i}).$$

Proof (Proof of the Corollary). From (2.1) we deduce $f_\mathrm{i} = 2t - f_s/2$ and $e = t + n_\mathrm{i} - f_\mathrm{s}/2 + 1$. Also, $n_\mathrm{s} = f_\mathrm{s}/2 + 2$. Using (4.2) and the proposition, we get

$$\Delta = \sum_{\ell \ge 0} \Delta_\ell \le C' \,(2e + f_\mathrm{i} + n_\mathrm{s}),$$

from which the assertion follows (the coefficient of f_s is negative and the additive constants can be bounded since $1 \le t$).

The proof of Proposition 4.12 will take up most of this subsection. We proceed as follows:

- Bound Δ_ℓ in terms of the number of edges in the hemispheres of nodes which are split (Lemma 4.14).

- Two terms appear: the first for the internal edges in the hemispheres, the other for the external edges (Lemma 4.16).
- Bound the first term with the counters a_ℓ, b_ℓ, f'_ℓ (Lemma 4.16 and Lemma 4.17).
- Bound the second term with the first term and the number of external nodes n_s (Lemma 4.18).

We decompose $\Delta_\ell = \Delta_{\ell,0} + \Delta_{\ell,1} + \Delta_{\ell,2}$, where the first term comes from the sweep C2→C1 at iteration ℓ, and the second from C1→C0. The third term coming from the sweep C0→external only decreases internal nodes and edges, so $\Delta_{\ell,2} \leq 0$ and we do not take that into account.

We define $\mathcal{L}_\ell = \{n : d(n) = \ell\}$, and, after having performed C2→C1 we define $\mathcal{M}_\ell$ as the set of those nodes of $\mathcal{L}_\ell$ which have not been split, as well as the children of those which have been split.

Lemma 4.14. *We have the bounds*

$$\Delta_{\ell,0} \leq 2 \sum_{n_* \in \mathcal{L}_\ell} \#\,(\textit{internal edges in } \mathcal{I}(n_*))\,,$$

$$\Delta_{\ell,1} \leq 2 \sum_{n_* \in \mathcal{M}_\ell} \#\,(\textit{internal edges in } \mathcal{I}(n_*))\,.$$

Proof. Cutting along a path γ adds $|\gamma| - 1$ internal edges to the triangulation (see Lemma 4.6).

The extension of the path $\tilde{\gamma}_\mathbf{S}$ to a splitting path $\gamma_\mathbf{S}$ adds at most 2 to its length (see step 7 for the case C2→ C1, and step 6 for the case C1→C0). Therefore, $|\gamma_\mathbf{S}| - 1 \leq 2|\tilde{\gamma}_\mathbf{S}|$, since $|\tilde{\gamma}_\mathbf{S}| \geq 1$.

For a given node n_*, all the paths $\tilde{\gamma}_\mathbf{S}$ are drawn in its hemisphere $\mathcal{I}(n_*)$ and each edge of this hemisphere is used in at most 1 path $\tilde{\gamma}_\mathbf{S}$ (see Theorem 4.9 and 4.10) so that for a given n_*, we have

$$\sum_\mathbf{S} |\tilde{\gamma}_\mathbf{S}| \leq \#\,(\text{internal edges in } \mathcal{I}(n_*))\,.$$

Summing over all splittable nodes yields the claim. (This is the crucial bound, which has become possible through our careful cutting procedures.)

We next bound the number of internal edges in the $\mathcal{I}(n_*)$.

Definition 4.15. *We define the numbers $a_{\ell,i}$ depending on whether we are before the C2→C1 sweep ($i = 0$) or after it but before the C1→C0 sweep ($i = 1$), and after that sweep ($i = 2$), all at iteration ℓ. The numbers $a_{\ell,i}$ are defined as the number of edges (x, y) with $d(x) = \ell$ and $d(y) = \ell + 1$, at iteration ℓ and before the sweep determined by i. Analogously, we define $b_{\ell,i}$ and $f'_{\ell,i}$.*

We also let $\hat{a}_{\ell,0}$ be the number of edges at the beginning of iteration ℓ with $d(x) = \ell - 1$ and $d(y) = \ell$. (This is not the same as $a_{\ell-1,0}$, which is defined for iteration $\ell - 1$.) Also, $\hat{a}_{\ell,1}$ is defined after sweep C2→C1.

Note that these numbers change with i, and depend on ℓ, since new edges are being added.

Lemma 4.16.

$$\Delta_{\ell,0} \le 6a_{\ell,0} + 12b_{\ell,0} + 6\hat{a}_{\ell,0} + 2 \sum_{n_* \in \mathcal{L}_\ell} (|\mathcal{E}(n_*)| - 3), \tag{4.4a}$$

$$\Delta_{\ell,1} \le 6a_{\ell,1} + 12b_{\ell,1} + 6\hat{a}_{\ell,1} + 2 \sum_{n_* \in \mathcal{M}_\ell} (|\mathcal{E}(n_*)| - 3). \tag{4.4b}$$

Proof. Starting from the relations of Lemma 4.14, we use Lemma 3.1 for the hemisphere of n_* which is a polygon with $p = |\mathcal{E}(n_*)|$ sides. Summing over $\mathcal{L}_\ell$ or $\mathcal{M}_\ell$, and since the only nodes x present in the hemisphere $\mathcal{I}(n_*)$ of a node n_* with $d(n_*) = \ell$ are such that $d(x) \in \{\ell-1, \ell, \ell+1\}$, the assertion follows. The factor $12 = 2 \cdot 6$ takes into account that the edge (n_*, m_*) can appear both in $\mathcal{I}(n_*)$ and in $\mathcal{I}(m_*)$ (if n_* and m_* are in $\mathcal{L}_\ell$ or $\mathcal{M}_\ell$.

So there remains to bound the terms on the r.h.s. of (4.4) in terms of the counters of the initial triangulation. This is done in the next lemmas.

The effect of the sweeps C2→C1 →C0 at iteration ℓ is summarized by

Lemma 4.17. *One has, for each* $\ell \ge 0$*:*

$$a_{\ell,0} = a_\ell, \quad b_{\ell,0} = b_\ell, \quad f'_{\ell,0} = f'_\ell, \tag{4.5a}$$

$$a_{\ell,1} \le a_{\ell,0} + 2f'_{\ell,0}, \quad a_{\ell,2} \le a_{\ell,1} + 2f'_{\ell,1}, \tag{4.5b}$$

$$f'_{\ell,1} \le 7f'_{\ell,0}, \tag{4.5c}$$

$$\hat{a}_{\ell,0} \le a_{\ell-1} + 16f'_{\ell-1}. \tag{4.5d}$$

Proof (Proof of Lemma 4.17). Proof of (4.5a). Let $\ell' < \ell$. During the iteration $\ell' < \ell$, we split nodes n_* of $\mathcal{L}_{\ell'}$ and $\mathcal{M}_{\ell'}$ into $n_{*,\mathbf{S}}$ with $d(n_{*,\mathbf{S}}) = \ell'$. This implies that every internal edge added during the iteration ℓ' has an end $n_{*,\mathbf{S}}$ at $d(n_{*,\mathbf{S}}) = \ell' < \ell$. But $a_{\ell,0}$ and $b_{\ell,0}$ count internal edges with both ends at depth $d(x) \ge \ell > \ell'$. The same argument also holds for $f'_{\ell,0}$. This means that at the beginning of the iteration ℓ, the values of the counters $a_{\ell,0}, b_{\ell,0}$ and $f'_{\ell,0}$ are still identical to the constants of the original triangulation.

Proof of (4.5b). We need to bound the added number of internal edges $(n_{*,\mathbf{S}}, y)$ in the sweep C2→C1 with $n_* \in \mathcal{L}_\ell$, $y \in \mathcal{I}(n_*)$, and $d(y) = \ell + 1$. By construction, this number is bounded by the number of paths $\gamma_\mathbf{S}$ which go through such a node y in the 2d triangulation $\mathcal{I}(n_*)$. Furthermore, by Theorem 4.9, each edge of $\mathcal{I}(n_*)$ is used in at most one path $\gamma_\mathbf{S}$. We deduce that, for two such nodes n_* and y, the number of added internal edges of type $(n_{*,\mathbf{S}}, y)$ is bounded by the degree of the edge (n_*, y) (the number of faces containing the edge). Summing these degrees for all such edges (n_*, y) is bounded by $2f'_{\ell,0}$. The same argument proves the second relation.

Proof of (4.5c). To prove the first relation, we need to bound the added number of internal faces $(n_{*,\mathbf{S}}, y, z)$ in the sweep C2→C1 at iteration ℓ when $n_{*,\mathbf{S}}$ is obtained from splitting some $n_* \in \mathcal{L}_\ell$ and y is such that $d(y) = \ell + 1$. But each added internal face $(n_{*,\mathbf{S}}, y, z)$ requires the addition of the internal edge $(n_{*,\mathbf{S}}, y)$. Furthermore, by definition of the move split-a-node-along-a-path, each new internal edge is added along with three internal faces. We deduce from (4.5b) that $f'_{\ell,1} - f'_{\ell,0} \le 3(a_{\ell,1} - a_{\ell,0}) \le 6f'_{\ell,0}$.

Proof of (4.5d). The reason this proof is tricky is that during the previous iteration $\ell - 1$, new internal edges satisfying $d(x) = \ell - 1$ and $d(y) = \ell$ have been added: the proof of this relation at iteration ℓ therefore involves the other relations of Lemma 4.17 at iteration $\ell - 1$.

When $\ell = 0$, (4.5d) obviously holds since both sides are 0. So we now assume $\ell > 0$ and that the conclusion of Lemma 4.17 holds for $\ell - 1$. We know that the sweep

C0$\rightarrow$external does not add any internal edge. From this we deduce that $\hat{a}_{\ell,0} \leq a_{\ell-1,2}$. The inequality (4.5d) then follows from the relations of Lemma 4.17 at iteration $\ell - 1$ we have already proved:

$$\begin{aligned}\hat{a}_{\ell,0} \leq a_{\ell-1,2} &\leq a_{\ell-1,1} + 2f'_{\ell-1,1}\\ &\leq a_{\ell-1,0} + 2f'_{\ell-1,0} + 14f'_{\ell-1,0}\\ &\leq a_{\ell-1} + 16f'_{\ell-1},\end{aligned}$$

which is the bound we seek.

We finally need to discuss the terms $|\mathcal{E}(n_*)| - 3$ of Lemma 4.16.

Lemma 4.18. *One has for each $\ell \geq 0$:*

$$\sum_{n_* \in \mathcal{L}_\ell} (|\mathcal{E}(n_*)| - 3) \leq \delta_{\ell,0} \cdot 3n_{\mathrm{s}},$$
$$\sum_{n_* \in \mathcal{M}_\ell} (|\mathcal{E}(n_*)| - 3) \leq \delta_{\ell,0} \cdot 3n_{\mathrm{s}} + \Delta_{\ell,0}.$$

Proof. The external degree of n_* is always 3 for those nodes which have been promoted to the surface by removing a tetrahedron (those which were at the surface at level $\ell = 0$ can of course have higher degree). Since the only way to lower the depth of a node is by removing a tetrahedron, we get

$$\begin{aligned}\sum_{n_* \in \mathcal{L}_\ell} (|\mathcal{E}(n_*)| - 3) &= \delta_{\ell,0} \sum_{n_* \in \mathcal{L}_0} (|\mathcal{E}(n_*)| - 3)\\ &= \delta_{\ell,0} \cdot (2e_{\mathrm{s}} - 3n_{\mathrm{s}})\\ &\leq \delta_{\ell,0} \cdot 3n_{\mathrm{s}},\end{aligned}$$

since the set $\mathcal{L}_0$ is the set of all external nodes.

The sum over $\mathcal{M}_\ell$ is more delicate, since the external degree of a node $n_* \in \mathcal{M}_\ell$ can be larger than 3. However, if we split a node $n_{*,\mathbf{S}}$ into $n_{*,\mathbf{S}\mathrm{R}}$ and $n_{*,\mathbf{S}\mathrm{L}}$, then the external degrees satisfy

$$(|\mathcal{E}(n_{*,\mathbf{S}\mathrm{L}})| - 3) + (|\mathcal{E}(n_{*,\mathbf{S}\mathrm{R}})| - 3) = (|\mathcal{E}(n_{*,\mathbf{S}})| - 3) + 1. \tag{4.6}$$

We know that nodes of $\mathcal{M}_\ell$ are the children of nodes in $\mathcal{L}_\ell$. Each node n_* of $\mathcal{L}_\ell$ is split along a binary tree into a set $\{n_{*,\mathbf{S}}\}_{\mathbf{S}}$ (some nodes of $\mathcal{M}_\ell$ are nodes of $\mathcal{L}_\ell$ which were not split; in this case the binary tree has no vertices). What (4.6) means is that each vertex of this binary tree adds 1 to the sum $\sum_{n_* \in \mathcal{M}_\ell} (|\mathcal{E}(n_*)| - 3)$. Therefore,

$$\sum_{n_* \in \mathcal{M}_\ell} (|\mathcal{E}(n_*)| - 3) \leq \sum_{n_* \in \mathcal{L}_\ell} (|\mathcal{E}(n_*)| - 3) + \#\,(\text{splits in C2}\rightarrow\text{C1})\,.$$

Since each split adds at least one internal edge, we have a bound

$$\sum_{n_* \in \mathcal{M}_\ell} (|\mathcal{E}(n_*)| - 3) \leq \sum_{n_* \in \mathcal{L}_\ell} (|\mathcal{E}(n_*)| - 3) + \Delta_{\ell,0}.$$

Proof (Proof of Proposition 4.12). We start from Lemma 4.16. The internal edges at the current point of the construction are all the edges with a node with $d(\cdot) = \ell$. These come in three types: Those connecting $(\ell, \ell + 1)$ (the $a_{\ell,0}$) those connecting (ℓ, ℓ) (the $b_{\ell,0}$), and those between $(\ell, \ell - 1)$ (the $\hat{a}_\ell$). Note that the last of these quantities is *not* equal to $a_{\ell-1}$ because the number of edges $(\ell, \ell - 1)$ might have changed in iteration $\ell - 1$. Lemma 4.17 provides, however, the necessary bounds.

Using (4.5a), (4.5d), and Lemma 4.18, we immediately get the bound on $\Delta_{\ell,0}$ we seek:

$$\begin{aligned}\Delta_{\ell,0} &\le 6a_\ell + 12b_\ell + 6(a_{\ell-1} + 16 f'_{\ell-1}) + 6n_s \cdot \delta_{\ell,0} \\ &\le 96(a_\ell + b_\ell + a_{\ell-1} + f'_{\ell-1} + n_s \cdot \delta_{\ell,0}). \end{aligned} \tag{4.7}$$

All internal edges added during the sweep C2→C1 at iteration ℓ have by construction an end x with $d(x) = \ell$. Therefore, the total number of internal edges added during the sweep C2→C1 at iteration ℓ, $\Delta_{\ell,0}$, is also given by

$$\left(a_{\ell,1} + b_{\ell,1} + \hat{a}_{\ell,1}\right) - \left(a_{\ell,0} + b_{\ell,0} + \hat{a}_{\ell,0}\right) = \Delta_{\ell,0}.$$

Using this relation and Lemma 4.18, starting from Lemma 4.16, we have:

$$\begin{aligned}\Delta_{\ell,1} &\le 12(a_{\ell,1} + b_{\ell,1} + \hat{a}_{\ell,1}) + \delta_{\ell,0} \cdot 6n_s + 2\Delta_{\ell,0} \\ &\le 12(a_{\ell,0} + b_{\ell,0} + \hat{a}_{\ell,0} + \Delta_{\ell,0}) + \delta_{\ell,0} \cdot 6n_s + 2\Delta_{\ell,0}.\end{aligned}$$

Using Lemma 4.17, we get:

$$\Delta_\ell = \Delta_{\ell,0} + \Delta_{\ell,1} \le 12(a_\ell + b_\ell + a_{\ell-1} + 16 f'_{\ell-1}) + 15\Delta_{\ell,0} + 6n_s \cdot \delta_{\ell,0}.$$

Replacing $\Delta_{\ell,0}$ by (4.7) yields the result we seek.

4.4. Reducing a triangulation with no internal nodes into a set of nuclei. Let T be any triangulation. In the previous section, we described an algorithm which transforms T into a new triangulation T' with no internal nodes. We now systematically apply the moves cut-a-3-face and open-a-2-face on every internal face of T' with less than 2 internal edges. We end up with a collection of triangulations $\{N_i\}$ satisfying the following properties:

– All nodes of any such N_i are external.
– All internal faces of any such N_i have at least 2 internal edges.

Any triangulation satisfying these two conditions is called a *nucleus*.

5. Part II: Bounding the Number of Triangulations

We showed that any triangulation can be reduced into a collection of nuclei using four moves. For the moment, we proceed without using the move cut-a-3-face. This implies that any triangulation can be transformed into a "tree of nuclei" (the formal definition of a tree of nuclei will be given later on) using the three remaining moves. Equivalently, this shows that any triangulation can be constructed from a tree of nuclei, using the inverse of these three moves. Bounding the number of trees of nuclei, and then bounding the number of ways one can perform the inverse moves on such a tree yields a bound on the total number of triangulations.

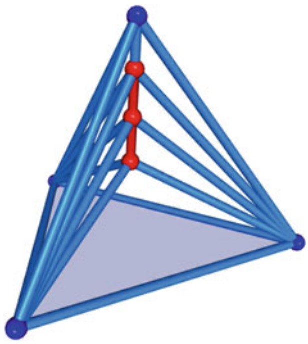

Fig. 9. The Christmas tree with $m = 3$ internal nodes. This triangulation can be rooted in more than one way

5.1. Rooted triangulations. We define what we mean by a rooted triangulation T and we show that one can label all external nodes of T. In the sequel, we use a particular labeling described below.

Definition 5.1. *A* rooted triangulation (T, F) *of the 3-ball is a triangulation T with one labeled external face F. This labeled face is called the* root. *The three nodes of the root are always labeled 0, 1, and 2.*

Remark 5.2. We will only consider rooted triangulations. This means for instance that talking about the Christmas tree T_m, $m > 1$ makes no sense, since there is more than one such rooted triangulation (Fig. 9). The exceptions are of course symmetric triangulations T such as the tetrahedron.

Definition 5.3. *Consider the boundary of a rooted triangulation* (T, F) *and let* $\mathcal{N}_\text{s}$ *be the set of all external nodes. We define a particular labeling* $h(\cdot) : \mathcal{N}_\text{s} \mapsto \mathbb{N} \cup \{0\}$ *of all external nodes.*

The labeling is defined as follows: the root is labeled as $(0, 1, 2)$. Any labeled edge can be seen as an element $(a, b) \in \mathbb{Z}_+^2$ with $a < b$.[7] We consider the lexical order on $\mathbb{Z}_+^2$. We start with the node 0. Its external flower is a 1d triangulation of the circle S^1 and it contains the edge $(1, 2)$ by definition. This edge determines the direction in which we label all unlabeled nodes of the external flower of node 0.

Next, we consider the external flower of node 1 and we look for the smallest labeled edge in the sense of the above ordering. In this case, this edge is $(0, 2)$. This edge fixes the direction in which we label all the yet unlabeled nodes of the external flower of node number 1. Notice that every labeled node is part of a face along with 2 already labeled nodes. This implies that the external flower of any labeled node contains a smallest labeled edge and as such can be directed.

We continue with all the nodes in their natural order until all external nodes of T are labeled.

5.2. Trees of nuclei. Since we work with rooted triangulations, from now on, we will only use rooted nuclei, namely:

Definition 5.4. *A* nucleus *is a* ***rooted*** *triangulation with no internal nodes such that every internal face has at most one external edge.*

[7] We use the notation $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

5.2.1. Rooted trees of nuclei and planar rooted trees. Let $\mathcal{N}$ be the set of all nuclei and $\mathcal{N}_{t,f}$ be the subset of all nuclei with t tetrahedra and f external faces.

Definition 5.5. *A rooted triangulation T is called a* rooted tree of nuclei *if all nodes of T are external and all internal faces of T have 0, 2, or 3 internal edges. (In other words, no internal face has 2 external edges.)*

In other words, a rooted tree of nuclei is simply a rooted triangulation which is obtained by gluing sequentially nuclei along pairs of their external faces. This is done in such a way that each nucleus is glued to an external face (a, b, c) of its parent through its root; 0 is identified with a, 1 with b and 2 with c. Once the tree is built, the external nodes are renumbered in the sense of Definition 5.3.

Since all external faces of a rooted triangulation are ordered, this defines a bijection between rooted trees of nuclei (T, F) and rooted planar trees with colored vertices in the following manner:

- Each nucleus of the triangulation (T, F) is represented by a colored vertex.
- The root-vertex of the planar tree represents the nucleus with the root F, *i.e.*, with the face $(0, 1, 2)$.
- Each internal face of the triangulation with three external edges is shared by two nuclei and hence it is represented in the tree by an edge connecting the corresponding two colored vertices.
- Since the internal faces with three external edges are ordered, this induces an order of the edges of the planar tree, say from left to right.

5.2.2. Hypothesis on the number of rooted nuclei. We next show how the question of exponential growth can be reformulated. We show that if there are not "too many" different types of nuclei, then there is indeed an exponential bound on the number of triangulations, when expressed in terms of the number of tetrahedra.

Hypothesis 5.6. There is a finite constant $K_1 > 1$ such that the number $\varrho(t, f_\mathrm{s})$ of face-rooted nuclei with f-vector $\langle t, f_\mathrm{s}, 0\rangle$ is bounded by K_1^t.

In order to alleviate the notation, from now on, we will denote f_s by f.

Lemma 5.7. *For any nucleus $N \in \mathcal{N}_{t,f}$ one has $f \le t + 3$.*

Proof. If N is a tetrahedron, the assertion is obvious. If N is non-trivial each tetrahedron of N can have at most 1 external face, since otherwise it would have an internal face with more than one external edge.

5.2.3. The number of rooted trees of nuclei. We use the classical method for counting planar ordered trees, generalized to the case of a multitude of different nodes, which are the face-rooted nuclei.

Definition 5.8. *Let $A_{v,t,f}$ be the number of rooted trees of nuclei with $v > 0$ nuclei, t tetrahedra and f external faces. We define $A_{0,t,f} = \delta_{t,0}\,\delta_{f,0}$.*

Our main bound is:

Proposition 5.9. *Under Hypothesis 5.6 there is a K_2, with $2 < K_2 < \infty$ such that for all t, f, one has*

$$\sum_v A_{v,t,f} \le K_2^t.$$

Proof. Consider a tree of nuclei, and let N be the nucleus containing the root F and assume that $N \in \mathcal{N}_{t_0, f_0}$. Removing N from the tree leads to $f_0 - 1$ rooted trees of nuclei, some of which may be empty. We let v_i, t_i, and f_i denote the counters for the branch i. Note that if a branch i has 0 nuclei, *i.e.*, if $v_i = 0$, then, obviously, $t_i = f_i = 0$. Thus, we get the relations:

$$\sum_{i=1}^{f_0-1} v_i = v - 1, \quad \sum_{i=1}^{f_0-1} t_i = t - t_0, \quad \sum_{i=1}^{f_0-1} \left(\delta_{f_i>0}(f_i - 1) + \delta_{f_i=0}\right) = f - 1. \tag{5.1}$$

In the sequel, we denote by $\sum'_{v,t,f,t_0,f_0}$ the sum over the set

$$\left\{v_i, t_i, f_i \;\middle|\; i = 1, \dots, f_0 - 1 \;, v_i \geq 0, t_i \geq 0, f_i \geq 0 \text{ and satisfying (5.1)}\right\}.$$

This observation allows us to write a recursive relation

$$A_{v,t,f} = \delta_{v,0}\,\delta_{t,0}\,\delta_{f,0} + \sum_{t_0>0,\, f_0\geq 4} \varrho(t_0, f_0) \sum\nolimits'_{v,t,f,t_0,f_0} \prod_{i=1}^{f_0-1} A_{v_i,t_i,f_i}. \tag{5.2}$$

Fix $M \in \mathbb{Z}_+$, and assume that v, t, f satisfy $3v + 3t + f \leq M$. By (5.1), we deduce

$$3v_i + 3t_i + f_i \leq 3v - 3 + 3t - 3t_0 + f \leq M - 1.$$

We define

$$A_M(s) = \sum_{3v+3t+f\leq M} A_{v,t,f}\, s^{3v+3t+f}.$$

Clearly, $A_0(s) = 1$ for all s, $A_M(0) = 1$ for all $M \geq 0$, and for a fixed s, $A_M(s)$ is an increasing sequence in M.

Multiplying (5.2) by $s^{3v+3t+f}$ and summing, we get, using (5.1):

$$\begin{aligned} A_M(s) = 1 + &\sum_{3v+3t+f\leq M} \sum_{t_0=1}^{t} \sum_{f_0=4}^{f} \varrho(t_0, f_0)\, s^{3+3t_0+1-\sum_{i=1}^{f_0-1}(\delta_{f_i>0}-\delta_{f_i=0})} \\ &\times \sum\nolimits'_{v,t,f,t_0,f_0} \prod_{i=1}^{\ell} A_{v_i,t_i,f_i}\, s^{3v_i+3t_i+f_i}. \end{aligned} \tag{5.3}$$

Using Lemma 5.7, we have

$$\begin{aligned} 3 + 3t_0 + 1 - \sum_{i=1}^{f_0-1}(\delta_{f_i>0} - \delta_{f_i=0}) &\geq 3 + 3t_0 + 1 - (f_0 - 1)\cdot 1 + 0 \\ &\geq 5 + 3t_0 - f_0 = 2(t_0 + 3 - f_0) + t_0 + f_0 - 1 \\ &\geq t_0 + f_0 - 1. \end{aligned}$$

Restricting to $0 \leq s \leq 1$, this implies

$$s^{3+3t_0+1-\sum_{i=1}^{f_0-1}(\delta_{f_i>0}-\delta_{f_i=0})} \leq s^{t_0+f_0-1}. \tag{5.4}$$

Using now Hypothesis 5.6, *i.e.*, $\varrho(t, f) \le K_1^t$, we get from (5.3) and (5.4):

$$\begin{aligned} A_M(s) - A_M(0) &\le \sum_{t_0=0}^{M} (sK_1)^{t_0} \sum_{f_0-1=0}^{M} \prod_{i=1}^{f_0-1} sA_{M-1}(s) \\ &\le \frac{1-(sK_1)^{M+1}}{1-sK_1} \frac{1-(sA_{M-1}(s))^{M+1}}{1-(sA_{M-1}(s))}. \end{aligned}$$

Restricting s further to $s \le 1/(2K_1)$ this leads to

$$A_M(s) - A_M(0) \le 2\frac{1-(sA_{M-1}(s))^{M+1}}{1-(sA_{M-1}(s))}.$$

Fix $s^* = \min(0.1, 1/(2K_1))$ and consider the map $F : x \mapsto 1 + 2/(1 - s^* \cdot x)$. One easily checks that F maps the interval $[1, 5]$ to itself. Furthermore, we have $s^* \cdot x \le 1$ for $x \in [1, 5]$. Starting with $x = A_0(s^*) = 1$ we conclude that for all M one has $A_M(s^*) \le 5$. This implies that the monotone sequence $A_M(s^*)$ converges as $M \to \infty$ and thus

$$A_{v,t,f} \le 5 \cdot (s^*)^{-3v-3t-f}.$$

Summing over v and using $v \le t$ and $f \le 4t$ we complete the proof.

5.3. Bound on triangulations. Having discussed the number of trees, we now study the number of ways these trees can be made into triangulations by identifying faces and nodes. This process is patterned after the work of [8] and [5].

Our bounds are based on using the inverses of the moves open-a-2-face, remove-1-tetra, and split-a-node-along-a-path. Since we are only interested in the bound, we will allow for inverse moves which do not necessarily lead to 3-balls.

Remark 5.10. While we over-count the number of triangulations, by allowing for moves which may not lead to 3-balls, we can in fact formulate precise conditions which guarantee that after each move, a 3-ball is obtained. These conditions are spelled out in Lemmas 5.11 and 5.14. This actually allows for efficient programming of the inverse operations.

5.3.1. Bounding the number of rooted triangulations with no internal nodes. Let $\mathcal{R}_{t,f}$ be the set of all rooted trees of nuclei with t tetrahedra and f external faces and let $\mathcal{T}_{t,f,0}$ be the set of all rooted triangulations with t tetrahedra, f external faces and no internal nodes. In this section, we will define the inverse move of open-a-2-face and we will use it to count the number of rooted triangulations with no internal nodes.

The inverse operation of open-a-face, which we will simply call *identification* when there is no ambiguity, is to identify two adjacent external faces, satisfying some conditions. Indeed, identifying any two adjacent external faces might lead to a complex which is not a triangulation. For instance, assume that (n_1, n_2, m_1) and (n_1, n_2, m_2) are two adjacent external faces such that there exists a node x adjacent to both m_1 and m_2. After identifying the two faces, we obtain a complex with a double edge $(x, m_1) = (x, m_2)$.

Lemma 5.11. *Consider a triangulation T. Let (a, b) be an external edge and let x, y be its two opposite external nodes (so that (x, a, b) and (y, a, b) are external faces). Assume that the following conditions are satisfied:*

– *The nodes x and y are not connected by an edge.*
– *The only nodes m such that (m, x) and (m, y) are edges are the two nodes a and b.*

Then, one can identify the two external nodes x and y as well as the two external faces sharing (a, b). This operation transforms a 3-ball to a 3-ball, and will be called identification (of two adjacent external faces).

Proof. The proof is left to the reader.

Proposition 5.12. *Under Hypothesis 5.6, there is a constant K_3 such that for all t and f one has*

$$|\mathcal{T}_{t,f,0}| \le K_3^t.$$

Proof. Let $T \in \mathcal{T}_{t,f,0}$ be any rooted triangulation with no internal nodes. Using repetitively the move open-a-2-face on T transforms it into a rooted triangulation T' with no internal nodes such that each internal face has 0, 1 or 3 external edges. In other words, T' is a rooted tree of nuclei. Equivalently, given a rooted tree of nuclei T' with t' tetrahedra and f' external faces, one can count the number of ways one can identify two adjacent external faces, *without any conditions guaranteeing ballness*. Summing this number over all rooted trees of nuclei gives us an upper bound on the number of rooted triangulations with no internal nodes.

We count the number of $T \in \mathcal{T}_{t,f,0}$ obtained by identification from a rooted tree of nuclei T' with t' tetrahedra and f' external faces. This means that we identify $D = (f' - f)/2$ pairs of adjacent external faces.

We first observe that choosing a pair of adjacent external faces is equivalent to choosing an external edge. We then note that some faces which are not adjacent in T' might become adjacent after some identifications are done. This means that we have a sequence $e_1, e_2, \ldots, e_\ell$ with $e_i \ge 1$ and $\sum_i e_i = D$ which is defined as follows:

– e_1 is the number of external edges (or equivalently of pairs of adjacent external faces) of T' which are identified.
– e_2 is the number of pairs of faces which were not adjacent in T' but became so after the first series of e_1 identifications. However, each identification of two adjacent external faces creates exactly two new pairs of adjacent external faces, implying that $e_2 \le 2e_1$.
– e_i is defined by analogy from the e_{i-1} identifications, implying that $e_i \le 2e_{i-1}$.

This leads to the following bound (recall that the number of external faces f' is bounded by $4t$):

$$|\mathcal{T}_{t,f,0}| \le \sum_{f'>f} |\mathcal{R}_{t,f'}| \sum_{\ell=1}^{D\equiv(f'-f)/2} \sum_{\sum_{i=1}^{\ell} e_i = D, e_i \ge 1} \binom{3f'/2}{e_1}\binom{2e_1}{e_2}\cdots\binom{2e_{\ell-1}}{e_\ell}.$$

Since $\binom{a}{b} \le 2^a$, we find, using Proposition 5.9 to bound $|\mathcal{R}_{t,f'}|$,

$$\begin{aligned}|\mathcal{T}_{t,f,0}| &\le \sum_{f'>f} |\mathcal{R}_{t,f'}| 2^{(5f'/2-f)} \sum_{\ell=1}^{D\equiv(f'-f)/2} \sum_{\sum_{i=1}^{\ell} e_i = D, e_i \ge 1} 1 \\ &\le \sum_{f'>f} |\mathcal{R}_{t,f'}| 2^{(5f'/2-f)} \sum_{\ell=1}^{D\equiv(f'-f)/2} \binom{D-1}{\ell-1}\end{aligned}$$

$$\leq \sum_{f'>f} |\mathcal{R}_{t,f'}| 2^{(3f'-3f/2)}$$

$$\leq \sum_{f'=f+2}^{4t} K_2^t K_2^{3f'}$$

$$\leq K_2^{13t} = K_3^t,$$

where $K_3 = K_2^{13}$.

The proof is complete.

5.3.2. Bounding the number of rooted triangulations (internal nodes included). In this section, we define the inverse moves of remove-1-tetra and split-a-node-along-a-path and we use them to count the number of rooted triangulations.

Definition 5.13. *We define the inverse move of remove-1-tetra, which we call* adding a tetrahedron: *Consider a triangulation T. Let x be an external node with external degree equal to 3 and let a_1, a_2 and a_3 be its external neighbors, i.e., (x, a_i) is an external edge. Adding a tetrahedron then consists in adding the face (a_1, a_2, a_3) and the tetrahedron (x, a_1, a_2, a_3).*

We define the inverse move of split-a-node-along-a-path.

Lemma 5.14. *Consider a triangulation T. Let $e = (a, b)$ be an external edge. Assume that the following conditions are satisfied:*

- *For each node m such that (m, a) and (m, b) are edges, (m, e) is a face.*
- *For each edge e' such that (e', a) and (e', b) are faces, (e', e) is a tetrahedron.*
- *There are no faces f such that (f, a) and (f, b) are both tetrahedra.*

Then, one can collapse the two nodes a and b, and the result is again a 3-ball. This move is called collapse of an external edge *or simply* collapse.

Proof. The proof is left to the reader.

Notation 5.15. *The above three conditions of a collapse are written in a concise way as*

$$\mathcal{I}(a) \cap \mathcal{I}(b) = \mathcal{I}(e).$$

In Sect. 4.3, we described an algorithm which transforms any triangulation with f-vector $\langle t, f, n\rangle$ into a triangulation with f-vector $\langle t', f', 0\rangle$. We have the following lemma:

Lemma 5.16. *There is a constant $K_4 > 0$ such that the f-vectors $\langle t, f_{\mathrm{s}}, n_{\mathrm{i}}\rangle$ and $\langle t', f'_{\mathrm{s}}, 0\rangle$ satisfy the following linear relation:*

$$t' \leq K_4 t, \quad f'_{\mathrm{s}} \leq K_4 t, \tag{5.5}$$

Proof. Let e, e' be the number of internal edges of both triangulations. By Corollary 4.13, we have $e' - e \leq C_\Delta(t + n_{\mathrm{i}})$. Using (4.1) and the obvious relations $n_{\mathrm{i}} \leq 4t$, $f_{\mathrm{s}} < 2n_{\mathrm{s}} < 4e_{\mathrm{i}}$ and $e_{\mathrm{i}} < 6t$, (and analogous ones with primes) the result follows.

This proves that any triangulation in $\mathcal{T}_{t,f,n}$ can be obtained from a triangulation with no internal nodes in $\mathcal{T}_{t',f',0}$ with a series of carefully chosen collapses and additions of tetrahedra, with t, f, n, t', f' satisfying (5.5).

We can now use a similar approach to that of the previous section. It is clear that choosing a triplet of external faces for the move add-1-tetrahedron is equivalent to choosing an external node x, and that choosing a couple of external nodes for collapse is equivalent to choosing an external edge.

5.4. Combining the bounds. Before we state our main result, we recall

Hypothesis 5.6. *There is a finite constant $K_1 > 1$ such that the number $\varrho(t, f)$ of face-rooted nuclei with f-vector $\langle t, f \rangle$ is bounded by K_1^t.*

Theorem 5.17. *Under Hypothesis 5.6 one has the bound: There is a finite constant C such that the number of rooted triangulations with f-vector $\langle t, f, n \rangle$ is bounded by*

$$|\mathcal{T}_{t,f,n}| \leq C^t. \tag{5.6}$$

Proof. Consider a rooted triangulation $T \in \mathcal{T}_{t,f,n}$ with t tetrahedra, f external faces and n internal nodes. We showed that T can be obtained from a rooted triangulation $T' \in \mathcal{T}_{t',f',0}$ by a series of carefully chosen collapses and additions of tetrahedra.

Note that the algorithm of Sect. 4.3 which transforms T into T' can always be stopped when the last internal node of T is removed. This implies that, in the inverse construction we are doing now, we must start by adding tetrahedra to T', and not by collapsing external edges. So the first step is to choose n_1 external nodes (of external degree 3) out of the $f'/2 + 2$ external nodes of T', and to insert a tetrahedron on each of them with one tip at the node. We call this "covering the node."

This reduces the number of external edges from $3f'/2$ to $3(f'/2 - n_1)$. Then, we choose m_1 external edges and we collapse them. The possibility to simultaneously collapse $m_1 > 1$ edges is justified as follows:

Any labeled triangulation can be viewed as a list of tetrahedra L_t satisfying certain conditions (a face is shared by no more than 2 tetrahedra etc…). From this point of view, collapsing an external edge e is simply the operation where we remove from L_t all the tetrahedra of $\mathcal{E}(e)$ (Lemma 5.14 guarantees that these conditions remain true, *i.e.*, that the resulting list of tetrahedra is indeed a triangulation). Let e_1 and e_2 be two collapsible edges. The construction implies that the order in which we collapse them is irrelevant and so, the idea that we simultaneously collapse m_1 edges makes sense. One should pay attention to the case where we collapse two edges $e_1 = (a, b_1)$ and $e_2 = (a, b_2)$ when $(b_1, b_2) = e_3$ is also an edge. In this case, all tetrahedra sharing one of the three edges are removed simultaneously. Clearly, this yields the same result regardless of the order in which we collapse e_1 and e_2.

The next step is to choose n_2 external nodes among the new possibilities which appear after performing the first series of coverings and collapses, and cover them. For each external edge e, we can associate four nodes: the two endpoints of e and the two nodes x_1, x_2 such that (x_i, e) is an external face. Assume that x is one of the n_2 chosen external nodes. The fact that x appeared after the first series implies that x is either one of the four nodes associated with one of the m_1 collapsed edges (note that these four nodes become three after the collapse), or that there is a node y among the first n_1 nodes such that (x, y) was an external edge (before covering y with a tetrahedron). But each such y has exactly 3 external neighbors. This implies that $n_2 \leq 3m_1 + 3n_1$ and the number of ways to choose these nodes is bounded by

$$\binom{3(m_1 + n_1)}{n_2}.$$

Continuing in this way, we choose m_2 external edges and we collapse them. Let e be such an edge. Again, e was not among the first m_1 edges. This implies that there must be a node x of the series of n_2 covered external nodes such that (e, x) formed an external face before covering x with a tetrahedron. But for each such x there are exactly three external edges satisfying this condition. We deduce that $m_2 \leq 3n_2$.

We continue adding tetrahedra and collapsing edges. This leads to two sequences $n_i, m_i, i = 1, \dots, \ell$, with $\ell \le n$, satisfying:

$$\begin{aligned} 1 \le n_i, \quad 0 \le m_i \le 3n_i, \quad \sum_{i=1}^{\ell} n_i = n, \\ 1 \le n_i \le 3n_{i-1} + 3m_{i-1}, \quad i > 1, \\ \sum_{i=1}^{\ell} (2n_i + 2m_i) + f = f'. \end{aligned} \tag{5.7}$$

The last identity follows because each collapse of an external edge and each covering of an external node (by a tetrahedron) reduces the number of external faces by 2. Note that some, or all, of the m_i's might be equal to zero. Using (5.7) we get a bound

$$\begin{aligned} |\mathcal{T}_{t,f,n}| \le \sum_{t',f'} |\mathcal{T}_{t',f',0}| \sum_{\ell=1}^{n} \sum_{\sum_{i=1}^{\ell} n_i = n, n_i \ge 1} \sum_{\sum_{i=1}^{\ell} m_i = (f'-f)/2 - n, m_i \ge 0} \\ \times \binom{f'/2+2}{n_1} \binom{3(n_1+m_1)}{n_2} \cdots \binom{3(n_{\ell-1}+m_{\ell-1})}{n_\ell} \\ \times \binom{3f'/2}{m_1} \binom{3n_1}{m_2} \cdots \binom{3n_{\ell-1}}{m_\ell}, \end{aligned}$$

where the sum over t', f' is restricted by (5.5). Bounding each binomial by a power of 2 and using Proposition 5.12, (5.7), (5.5), and $n_{\mathrm{s}} = f_{\mathrm{s}}/2 + 2$, we get, as in the proof of Proposition 5.12,

$$|\mathcal{T}_{t,f,n}| \le \sum_{t',f' \le K_4 t} K_3^{t'} \le C^t .$$

This shows (5.6) and completes the proof.

Acknowledgements. We profited from useful discussions with G. Ziegler. We thank the referee for a very careful reading of the manuscript. This work was partially supported by the Fonds National Suisse and by an ERC FP7 "Ideas" Advanced Grant.

References

1. Aharonov, E., Bouchbinder, E., Hentschel, H.G.E., Ilyin, V., Makedonska, N., Procaccia, I., Schupper, N.: Direct identification of the glass transition: Growing length scale and the onset of plasticity. Europhys. Lett. **77**, 56002 (2007)
2. Ambjørn, J., Durhuus, B., Jónsson, T.: Three-dimensional simplicial quantum gravity and generalized matrix models. Mod. Phys. Lett. A **6**, 1133–1146 (1991)
3. Ambjørn, J., Durhuus, B., Jónsson, T.: *Quantum geometry*. Cambridge Monographs on Mathematical Physics, Cambridge: Cambridge University Press, 1997
4. Aste, T., Sherrington, D.: Glass transition in self-organizing cellular patterns. J. Phys. A **32**, 7049–7056 (1999)
5. Benedetti, B., Ziegler, G.M.: On locally constructible spheres and balls. Acta Math. **206**, 205–243 (2011)
6. Bing, R.H.: *Some aspects of the topology of* 3*-manifolds related to the Poincaré conjecture*. In: Lectures on modern mathematics, Vol. **II**, New York: Wiley, 1964, pp. 93–128
7. Diestel, R.: *Graph theory*, Volume **173** of Graduate Texts in Mathematics, 4th edn. Heidelberg: Springer, 2010

8. Durhuus, B., Jónsson, T.: Remarks on the entropy of 3-manifolds. Nucl. Phys. B **445**, 182–192 (1995)
9. Eckmann, J.-P.: A topological glass. J. Stat. Phys. **129**, 289–309 (2007)
10. Eckmann, J.-P., Younan, M.: Decay of correlations in a topological glass. Phil. Mag. **92**, 98–119 (2012)
11. Furch, R.: Zur Grundlegung der kombinatorischen Topologie. Abhandlungen aus dem mathematischen Seminar der Universität Hamburg **3**, 69–88 (1924)
12. Goodrick, R.E.: Non-simplicially collapsible triangulations of I^n. Proc. Camb. Phil. Soc. **64**, 31–36 (1968)
13. Gromov, M.: Spaces and questions. Geom. Funct. Anal. **GAFA 2000**, 118–161 (2000)
14. Hachimori, M.: Simplicial complex library. http://infoshako.sk.tsukuba.ac.jp/~HACHI/math/library/index_eng.html, 2001
15. Magnasco, M.O.: Two-dimensional bubble rafts. Phil. Mag. B **65**, 895–920 (1992)
16. Pfeifle, J., Ziegler, G.M.: Many triangulated 3-spheres. Math. Ann. **330**, 829–837 (2004)
17. Tutte, W.T.: A census of planar triangulations. Canad. J. Math. **14**, 21–38 (1962)

Communicated by M. Salmhofer